Geometry and Topology Qualifying Exam, January 2013

1. Suppose that R is a rational function, i.e. $R(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials with $\deg(p) < \deg(q)$. Let $a_1, ..., a_n$ denote the distinct roots of $q(z) = 0$, and suppose that $p(a_i) \neq 0$, $i = 1, .., n$. Show that R has a partial fraction decomposition, i.e.

$$R(z) = \sum_{i=1}^{n} r_i(z),$$

where r_i is a rational function which vanishes at ∞ and whose only singularity is a pole at $z = a_i$. Hint: consider the Laurent expansion of R around a_i.

2. Find the curvature of the graph of the function $y = \cosh x$ at all points.

3. (a) Using calculus only, show that a one-form η on S^1 is exact if and only if

$$\int_{S^1} \eta = 0.$$

(b) Using the Mayer-Vietoris sequence applied to de Rham cohomology, show that a two-form ω on S^2 is exact if and only if

$$\int_{S^2} \omega = 0.$$

4. The Klein bottle can be defined as a square $[0, 1] \times [0, 1]$ with $(s, 0)$ identified with $(s, 1)$ and $(0, t)$ identified with $(1, 1 - t)$ for every $0 \leq s, t \leq 1$.

a) In terms of this realization, find a presentation for the fundamental group (at some base point).

b) Find the integral homology (homology with coefficients in $\mathbf{Z}$) of the Klein bottle.

5. In this problem you are proving a theorem in several steps. You are invited to provide an argument for each step, assuming the previous ones, even if you did not complete them.

Theorem: Let $f : \mathbf{S^2} \to \mathbf{R^2}$ be a continuous map. There exists an $x \in \mathbf{S^2}$ such that $f(x) = f(-x)$.

a) Suppose that no such x exists, so that $f(x) \neq f(-x)$ for all x. Use this to construct a continuous map $g : \mathbf{S^2} \to \mathbf{S^1}$ such that $g(x) = -g(-x)$ for all x.

b) Consider the restriction h of g to $\mathbf{S^1}$ thought of as the equator of $\mathbf{S^2}$. Prove that h is a map of an odd degree. Hint: you are welcome to assume that f (and hence also g and h) is a smooth function. An integral of dH from 0 to 2π (where H is the lift of h to the real line) can then be used to calculate the degree.

c) Remembering that g was defined on all of $\mathbf{S^2}$, show that, on the other hand, its restriction to $\mathbf{S^1}$ must be homotopic to a constant map.

d) Derive a contradiction from b) and c).

6. Let $H : \mathbf{R^2} \to \mathbf{R}$ be a $\mathbf{C^2}$-smooth function. Suppose that $p_t(x, y)$ is the solution of the system of ODE's

$$\frac{dp_t}{dt} = -\frac{\partial H}{\partial q}(p_t, q_t)$$
$$\frac{dq_t}{dt} = \frac{\partial H}{\partial p}(p_t, q_t)$$

with the initial condition $p_0 = x$, $q_0 = y$. Prove that the pullback of the area form $dp \wedge dq$ under the map $(x, y) \mapsto (p_t, q_t)$ equals $dx \wedge dy$ (i.e. that this map preserves the area form in the plane).

Hint: this is obviously true for $t = 0$. You can do the problem directly, differentiating the pullback of the area form in t or use known theorems, e.g. involving Lie derivative.