

Integration Workshop 2003

Algebra Exercises

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Vector Spaces and Linear Transformations

1. Let V be the set of all matrices $A \in M_{2 \times 2}(\mathbb{C})$ with trace 0.
 - (a) Show that V is a vector space over the field of *real* numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.
 - (b) Find a basis for this vector space.
 - (c) Let W be the set of all matrices $A \in V$ such that $a_{21} = -\bar{a}_{12}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W .

2. Consider $M_{n \times n}(F)$ as a vector space over a field F , and let $B \in M_{n \times n}(F)$ be fixed. If

$$T(A) = AB - BA,$$

verify that T is a linear transformation from $M_{n \times n}(F)$ into $M_{n \times n}(F)$.

3. Let V be the set of all complex numbers regarded as a vector space over \mathbb{R} (with the usual operations). Find a function $f : V \rightarrow V$ which is a linear transformation on V , but which is not a linear transformation on \mathbb{C} , i.e., which is not complex linear.
4. Let $B \in M_{2 \times 2}(\mathbb{C})$ be given by

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix},$$

and let T be the linear operator on $M_{2 \times 2}(\mathbb{C})$ (as a \mathbb{C} vector space) defined by $T(A) = BA$. What is the rank of T ? Can you describe T ?

5. Let V be a vector space over the field F and T a linear operator on V . If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T ? Give an example of a linear operator T on \mathbb{R}^2 such that $T^2 = 0$, but $T \neq 0$.
6. Let V be a finite-dimensional vector space and let T be a linear operator on V . Suppose that $\text{rank}(T^2) = \text{rank } T$. Prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.
7. Suppose that V , W_1 , and W_2 are subspaces of some larger vector space U over K . Prove or give a counterexample: $V \oplus W_1 = V \oplus W_2$ (as subspaces of U) implies that $W_1 = W_2$?
8. Let $P^n(F)$ denote the vector space of polynomials of degree at most n with coefficients in the field F . Let $b \in \mathbb{R}$. Define $T_b : P^2(\mathbb{R}) \rightarrow P^2(\mathbb{R}) : p(x) \mapsto p(x + b)$. Prove this is a linear transformation.
9. Let \mathbb{C} be the set of complex numbers and let \mathbb{R} be the field of real numbers. With the usual operations, \mathbb{C} is a vector space over \mathbb{R} . Describe explicitly an isomorphism of this space onto \mathbb{R}^2 .
10. Let \mathbb{R} be the set of real numbers, and let \mathbb{Q} be the field of rational numbers. Show that \mathbb{R} is a vector space over \mathbb{Q} .
11. Consider the space of column matrices $M_{n \times 1}(F)$ over a field F . Let $A \in M_{n \times n}(F)$; then A defines a linear operator L_A on $M_{n \times 1}(F)$ through left multiplication: $L_A(X) = AX$. Prove that every linear operator on $M_{n \times 1}(F)$ is left multiplication by some $n \times n$ matrix, i.e., is L_A for some A .

Now suppose V is an n -dimensional vector space over the field F , and let β be an ordered basis for V . For each $\alpha \in V$, define $U\alpha = [\alpha]_\beta$. Prove that U is an isomorphism of V onto $M_{n \times 1}(F)$. If T is a linear operator on V , then UTU^{-1} is a linear operator on $M_{n \times 1}(F)$. Accordingly, UTU^{-1} is left multiplication by some $n \times n$ matrix A . What is A ?
12. Choose a basis for $P^3(\mathbb{C})$ as a complex vector space, and give a matrix representation for D , the differential operator that sends a polynomial $p(x)$ to its derivative $p'(x)$. Identify the range and null space.

13. Let $A, B \in M_{n \times n}(\mathbb{C})$. Prove that if A and B are similar, then A^n and B^n are similar.

Polynomial Rings

- Let $U(R)$ be the group of units of R ; that is, $U(R)$ is the set of all elements with multiplicative inverses.
 - Show that $U(R)$ is a group.
 - Identify $U(\mathbb{Z})$.
 - Identify $U(\mathbb{Z}_n)$.
 - Identify $U(\mathbb{Z}_4[x])$.
 - Identify $U(\mathbb{Z}_n[x])$. (Hint: first assume $n = p^k$, p prime, $k \in \mathbb{N}$.)
- Prove the Remainder Theorem: Let R be a commutative ring with unity, and $f, g \in R[x]$. If g has leading coefficient b , then there exist $k \in \mathbb{N}$, and $q, r \in R[x]$ such that

$$b^k f = qg + r$$

with $\deg(r) < \deg(g)$. If b is not a zero divisor in R , then q and r are unique. If $b \in U(R)$ we may take $k = 0$.

- Let R be a commutative ring with unity, $f \in R[x]$, $a \in R$. Show that for $g(x) = x - a$ the remainder guaranteed by the Remainder Theorem is $f(a)$.
- Let F be a field. Show that the intersection of any number of ideals in $F[x]$ is an ideal.
 - Write a single monic generator for the ideal of polynomials $I \subseteq \mathbb{C}[x]$ which vanish at $x = 2$.
 - Write a single monic generator for the ideal of polynomials $J \subseteq \mathbb{C}[x]$ which vanish at $x = 2$ and $x = 5$.
 - Write a single monic generator for the ideal of polynomials $K \subseteq \mathbb{C}[x]$ generated by $x^2 + 3x + 2$ and $x^2 - 3x - 10$.
 - Write a single monic generator for the ideal of polynomials $L \subseteq \mathbb{C}[x]$ generated by $x^2 + 3x + 2$ and $x^2 - 3x + 2$.

Polynomials and Linear Algebra

1. Find characteristic and minimal polynomials for the following matrices.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Let n be a positive integer, and F a field. Suppose $A \in M_{n \times n}(F)$, and $P \in GL_{n \times n}(F)$. (That is, A is an $n \times n$ matrix over F , and P is an invertible $n \times n$ matrix over F .) If f is any polynomial over F , prove that

$$f(P^{-1}AP) = P^{-1}f(A)P.$$

3. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Find the characteristic polynomial $p(x)$ of A . Find a diagonal matrix D , and change of basis matrix Q such that $D = Q^{-1}AQ$. Compute $p(A)$ and $p(D)$.
4. Write a matrix $A \in M_{4 \times 4}(\mathbb{C})$ such that $(A - 3I)^2 = 0$, but $A - 3I \neq 0$. Write a matrix $A \in M_{4 \times 4}(\mathbb{C})$ such that $(A - 3I)^3 = 0$, but $(A - 3I)^2 \neq 0$.

Definition: Let K be a commutative ring with unity. We call a function $D : M_{2 \times 2}(K) \rightarrow K$ *bilinear* if D is a linear function of the i th row when the other row is held fixed. We say D is *alternating* if the following two conditions hold:

- a) $D(A) = 0$ whenever two rows of A are equal.
 - b) If A' is a matrix obtained from A by interchanging two rows of A , then $D(A') = -D(A)$.
5. Let K be a commutative ring with identity, and let $D : M_{2 \times 2}(K) \rightarrow K$ be an alternating bilinear function. Show that $D(A) = \det(A)D(I)$ for all A . Now use this result (no computations with the entries allowed) to show that $\det(AB) = \det(A)\det(B)$ for any $A, B \in M_{2 \times 2}(K)$.

Modules

1. Let M be a left R -module, and let $m \in M$. Show that $L_m := \{a \in R \mid am = 0\}$ is a left ideal of R .

2. Let G be a group. Consider the action of \mathbb{Z} on G : $n \cdot g = g^n$. When is G a \mathbb{Z} -module? (That is, what conditions must be placed on G so that it satisfies the properties of a \mathbb{Z} -module.)
3. Show that \mathbb{Q} is a \mathbb{Z} -module that is not free.
4. Consider the ideal $I := \langle 2 \rangle \subseteq \mathbb{Z}_6$.
 - (a) Show that we can consider I to be a \mathbb{Z}_6 -module. (That is, define the action of \mathbb{Z}_6 on I and show that the necessary properties are satisfied.)
 - (b) In the previous exercise, note that I and \mathbb{Z}_3 are isomorphic as groups. Under what conditions (on r and s) can \mathbb{Z}_r (with its natural group structure) be given the structure of a module over \mathbb{Z}_s (with its natural ring structure)?

Finitely Generated Modules over a PID

1. Let $R = \mathbb{C}[x]$, $M = \mathbb{C}[x, y]$. Let M be considered an R -module with scalar multiplication just normal polynomial multiplication.
 - a) Verify that M is indeed an R -module.
 - b) M is not isomorphic to $R/(r_1) \oplus \cdots \oplus R/(r_k) \oplus R^m$ for any m, r_1, \dots, r_k . Why does this not contradict our theorem? Can you describe the module M in a different way to make this more clear?
2. Let $\alpha, \beta \in \mathbb{C}$ be fixed. Let $R = \mathbb{C}[x, y]$, $M = \mathbb{C}$. We can consider M to be a R -module by defining the scalar multiplication for $f \in \mathbb{C}[x, y]$, $z \in \mathbb{C}$ to be $f \cdot z := f(\alpha, \beta) \cdot z$.
 - a) Verify that M is indeed an R -module.
 - b) M is not isomorphic to $R/(r_1) \oplus \cdots \oplus R/(r_k) \oplus R^m$ for any m, r_1, \dots, r_k . Why does this not contradict our theorem? Can you describe the module M in a different way to make this more clear? (Hint: Try writing M as a quotient of $\mathbb{C}[x, y]$.)
3. Let $R = \mathbb{Z}[x]$, $M = \mathbb{Z}_2$. We can consider M to be an R -module by defining the scalar multiplication for $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to be $f \cdot 1 := a_0 \pmod{2}$.

- (a) Verify that M is indeed an R -module.
- (b) M is not isomorphic to $R/(r_1) \oplus \cdots \oplus R/(r_k) \oplus R^m$ for any m, r_1, \dots, r_k . Why does this not contradict our theorem? Can you describe the module M in a different way to make this more clear? (Hint: Try writing M as a quotient of $\mathbb{Z}[x]$.)
4. Let F be a field, and V a finite dimensional F -vector space, and $T : V \rightarrow V$ a fixed linear transformation. Show that we can consider V as an $F[x]$ -module, with $f \cdot v = f(T)(v)$, which we shall denote V_T .
5. Suppose $W = F[x]\langle v \rangle$ is a cyclic submodule of V_T with annihilator (f) where $\deg(f) = k > 0$. Show that the set $\{v, Tv, T^2v, \dots, T^{k-1}v\}$ is a (vector space over F) basis for W . We will call v a *cyclic* vector for W .
6. Let F be a field and $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$ be a monic polynomial in $F[x]$. Define the companion matrix of f to be

$$C(f) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

Let T be the linear transformation associated with $C(f)$.

- a) Show that the minimal polynomial of T , $m_T(x) = f(x)$. (Hint: consider the action of T on the standard basis.)
- b) Show that the characteristic polynomial of a companion matrix $C(f)$ is $\pm f(x)$.

Duality

1. If $A, B \in M_{n \times n}(F)$, show that $\text{trace}(AB) = \text{trace}(BA)$. Now show that similar matrices have the same trace.
2. Let $p \in P^2(F)$. Define three linear functionals on $P^2(F)$ by

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.

3. If $A, B \in M_{n \times n}(\mathbb{C})$, show that $AB - BA = I$ is impossible.
4. Let m and n be positive integers and F a field. Let f_1, \dots, f_m be linear functionals on F^n . For $\alpha \in F^n$ define

$$T\alpha = (f_1(\alpha), \dots, f_m(\alpha)).$$

Show that T is a linear transformation from F^n into F^m . Then show that every linear transformation from F^n into F^m is of the above form, for some f_1, \dots, f_m .

5. Show that the trace functional on $M_{n \times n}(F)$ is unique in the following sense. If f is a linear functional on $M_{n \times n}(F)$ such that $f(AB) = f(BA)$ for each $A, B \in M_{n \times n}(F)$, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function.
6. Let S be a set, F a field, and $V(S; F)$ the space of all functions from S into F :

$$(f + g)(x) = f(x) + g(x),$$
$$(cf)(x) = cf(x)$$

Let W be any n -dimensional subspace of $V(S; F)$. Show that there exist points $x_1, \dots, x_n \in S$ and functions $f_1, \dots, f_n \in W$ such that $f_i(x_j) = \delta_{ij}$.

7. Consider the vector space $\mathbb{R}[x]$. Let $a, b \in \mathbb{R}$ be fixed, and let f be the linear functional on v defined by

$$f(p) = \int_a^b p(x) dx.$$

If D is the differentiation operator on $\mathbb{R}[x]$, what is D^*f ?

8. Let $B \in M_{n \times n}(F)$ be fixed, and define $T : M_{n \times n}(F) \rightarrow M_{n \times n}(F) : A \mapsto AB - BA$. If f is the trace function, what is T^*f ?

9. a) If $B \in M_{n \times n}(F)$ is fixed, define a function $f_B : M_{n \times n}(F) \rightarrow F$ by $f_B(A) = \text{trace}(B^t A)$. Show that f_B is a linear functional on $M_{n \times n}(F)$.
- b) Show that every linear functional on V is of the above form, i.e., is f_B for some B .
- c) Show that $\phi : M_{n \times n}(F) \rightarrow M_{n \times n}(F)^* : B \mapsto f_B$ is an isomorphism.

Inner Products

1. Let V be a vector space over F . Show that the sum of two inner products on V is an inner product on V . Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.
2. Let V be an inner product space, and let α, β be vectors in V . Show that $\alpha = \beta$ if and only if $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle$ for every $\gamma \in V$.
3. Find an inner product on \mathbb{R}^2 such that $\langle e_1, e_2 \rangle = 2$.