Integration Workshop 2003
Analysis Problems

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CALCULUS

- **Multivariable differential calculus**

1. Let f and $\partial f/\partial y$ be continuous on $[0,1]\times[0,1]$ and assume that $p,q:[0,1]\to[0,1]$ are differentiable. Define
$$F(y)=\int_{p(y)}^{q(y)} f(x,y)\,dx, \qquad y\in[0,1].$$
Use the chain rule to find $F'(y)$. *Hint.* Consider $G(x_1,x_2,x_3)=\int_{x_1}^{x_2} f(t,x_3)\,dt$.

2. Define $C^1(R^2,R^3)$ to be the continuously differentiable functions from R^2 to R^3 and $C_0^1(R^2,R^3)$ to be those functions that vanish at the origin. For $f,g\in C^1(R^2,R^3)$, define $f\sim g$ if
$$\lim_{x\to 0}\frac{f(x)-g(x)}{x}=0.$$
Show that this relation is an equivalence relation. Describe the algebraic and geometric properties of the spaces $C^1(R^2,R^3)/\sim$, $C_0^1(R^2,R^3)$, and $C_0^1(R^2,R^3)/\sim$.

3. A function $f:S\to R^n$ is *homogeneous of degree* p if
$$f(\lambda x)=\lambda^p f(x) \quad \text{for every } \lambda\in R \text{ and every } x\in S \text{ for which } \lambda x\in S.$$
If S is open and f is differentiable at x show that
$$x\cdot\nabla f(x)=pf(x).$$
Hint. Define $g(\lambda)=f(\lambda x)$ and compute $g'(1)$.
Conversely, show that if $x\cdot\nabla f(x)=pf(x)$ for all $x\in S$, then f is homogeneous of degree p.

4. Let S be an open connected set subset of R^n. Let $f:S\to R^n$ be differentiable at each point of S. If the total derivative vanishes for each $c\in S$, then f is constant on S.

5. let $GL(n)$ be the invertible matrices on R^n and define $\iota:GL(n)\to GL(n)$ by $\iota(M)=M^{-1}$. Find the total derivative of ι at A.

- **Implicit functions and extremum problems**

1. Show that the systems of equations

$$\begin{aligned} 3x+y-z+u^2 &= 0 \\ x-y+2z+u &= 0 \\ 2x+2y-3z+2u &= 0 \end{aligned}$$

   can be solved for x,y,u in terms of z; for x,z,u in terms of y; for y,z,u in terms of x; but not for x,y,z in terms of u.

2. Find the maximum of $(x_1\cdots x_n)^2$ under the restriction

$$x_1^2+\cdots+x_n^2=1.$$

   Use this to prove that the harmonic mean is less than the arithmetic mean.

$$(a_1\cdots a_n)^{1/n}\le \frac{1}{n}(a_1+\cdots+a_n).$$

3. State conditions of f and g that will ensure that the equations

$$x=f_1(u,v), \qquad y=f_2(u,v)$$

   can be solved for u and v in a neighborhood of (x_0,y_0). Call the solution

$$u=g_1(x,y), \qquad v=g_2(x,y)$$

   and let $J=\partial(f_1,f_2)/\partial(u,v)$. Show that

$$\frac{\partial g_1}{\partial x}=\frac{1}{J}\frac{\partial f_2}{\partial v}, \quad \frac{\partial g_1}{\partial y}=-\frac{1}{J}\frac{\partial f_1}{\partial v}, \quad \frac{\partial g_2}{\partial x}=-\frac{1}{J}\frac{\partial f_2}{\partial u}, \quad \frac{\partial g_2}{\partial y}=\frac{1}{J}\frac{\partial f_1}{\partial u},$$

4. Give a simpler formulation for the second derivative test for extrema if the domain of the function is a subset of the plane.

- **Multivariable Riemann integral**

For a random variable X, the *cumulative distribution function* $F_X(x) = P\{X \le x\}$. X is called continuous if F_X is differentiable. The derivative f is called the *density function*. Then $P\{X \in A\} = \int_A f(x)\, dx$. The *joint cumulative distribution function* for (X, Y),

$$F_{(X,Y)}(x, y) = P\{X \le x, Y \le y\}$$

is called continuous if there exists a density function $f_{(X,Y)}$ so that

$$F_{(X,Y)}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{(X,Y)}(u, v)\, du\, dv.$$

X and Y are called independent if $f_{(X,Y)}$ factors into a function of x and a function of y.

1. Let g be continuous and strictly increasing on the support of f_X. Find the density of $g(X)$. Generalize this.

2. Let X and Y be independent with densities f_X and f_Y. Find the density of f_{X+Y}.

3. Let X and Y be independent standard normal random varables, i.e., having density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty.$$

   Find the joint density for $(X/Y, X^2 + Y^2)$. Find the density for X/Y and for $X^2 + Y^2$. Generalize the second formula to n independent standard normals.

4. Let X and Y be independent exponential random variables, i.e., having density

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0.$$

   Find the joint density of $(X+Y, X/(X+Y))$. Find the density for $X+Y$ and for $X/(X+Y)$. Generalize this to n independent exponentials.

5. Let $f(x, y) = (x - y)/(x + y)^2$. Show that

$$\int_0^1 dx \int_0^1 f(x, y)\, dy = -\int_0^1 dy \int_0^1 f(x, y) dx = \frac{1}{2}$$

6. Show how the Stokes' formula and the divergence theorem follow from Stokes' theorem.

REAL ANALYSIS

• Sequences and series

1. Assume that $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \ldots$, and suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2. Let $\{a_n : n \geq 0\}$ be a strictly positive sequence of numbers and prove that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

   Let $a_n = n^n/n!$. Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = e.$$

3. Let $\{a_n : n \geq 0\}$ be a real-valued sequence of numbers and define $\bar{a}_n = (a_1 + \cdots + a_n)/n$. Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} \bar{a}_n \leq \limsup_{n\to\infty} \bar{a}_n \leq \limsup_{n\to\infty} a_n.$$

   What conculsions can you reach?

4. Define the sequence $\{f_n; n \geq 1\}$ recursively by $f_1 = f_2 = 1$ and $f_{n+2} = f_n + f_{n+1}$. Show that

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}.$$

   Hint. Show that $f_{n+2}f_n - f_{n+1}^2 = (-1)^{n+1}$ and deduce that $|f_{n+1}/f_n - f_{n+2}/f_{n+1}| < n^{-2}$ if $n > 4$.

5. Test for convergence (p, q and r are fixed real numbers.)

$$\sum_{n=1}^{\infty} n^k e_n \qquad \sum_{n=1}^{\infty} (\log n)^p \qquad \sum_{n=1}^{\infty} p^n n^p \qquad \sum_{n=1}^{\infty} \frac{1}{n^p - n^q} \qquad \sum_{n=1}^{\infty} \frac{1}{p^n - q^n},$$

$$\sum_{n=1}^{\infty} \frac{p^n}{n^q (\log n)^r}, \qquad \sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{(1+\frac{1}{n})}}.$$

6. Let a be a positive real number. Define a sequence $\{x_n : n \geq 0\}$ by $x_0 = 0$, $x_{n+1} = a + x_n^2$ $n \geq 0$. Find a necessary and sufficient condition on a in order that a finite limit $\lim_{n\to\infty} x_n$ exists.

7. Let a and b be fixed natural numbers, $a \geq b \geq 1$ and define

$$s_n = \sum_{k=an+1}^{bn} \frac{1}{k}.$$

   Prove that

$$\lim_{n\to\infty} s_n = \log \frac{b}{a}.$$

- **Sequences of Functions**

1. Let $f_n(x) = x^n$. Prove that the sequence $\{f_n : n \geq 0\}$ converges pointwise but not uniformly on [0,1]. Assume that g is continuous on [0,1] with $g(1) = 0$ and define $g_n(x) = x^n g(x)$. Prove that $\{g_n : n \geq 0\}$ converges uniformly on [0,1].

2. Let f be a continuous real valued function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x)$ exists (finitely). Prove that f is uniformly continuous.

3. Let f be a continuous function on $[0, 1]$. Evaluate the following limits:
$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx,$$
$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx.$$

4. Let $\{g_n : n \geq 0\}$ be continous functions on [0,1] and assume that
$$g_{n+1}(x) \leq g_n(x) \qquad \text{for each } x \in [0, 1].$$
Prove that
$$g(x) = \sum_{n=0}^{\infty} (-1)^n g_n(x)$$
converges uniformly on [0,1]. for all x. Prove that $g\colon \mathbb{R} \to \mathbb{R}$ is continuous.

5. Let $\{f_n\}$ be a sequence of continuous functions from $[0, 1]$ to $\mathbb{R}$. Suppose that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [0, 1]$, and also that for some constant K, we have
$$\left| \int_0^1 f_n(x)\, dx \right| \leq K < \infty$$
for all n. Does
$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0?$$

6. Assume that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2. Find the radius of convergence of
$$\sum_{n=0}^{\infty} a_n^k x^n, \qquad \sum_{n=0}^{\infty} a_n x^{kn} \qquad \sum_{n=0}^{\infty} a_n x^{n^2}.$$

7. Suppose the coefficients of the power series
$$\sum_{n=0}^{\infty} a_n x^n$$
are given by the recurrence relation $a_0 = 0, a_1 = -1$,
$$3a_n + 4a_{n-1} - a_{n-2} = 0, \qquad n = 2, 3, \cdots$$
Find the radius of convergence of the series and the function to which it converges in its interval of convergence.

- **First order ordinary differential equations**

1. Let $f(x)$ be a real valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is less than $1 + \frac{\pi}{4}$.

2. Let J_0 be an $n \times n$ matrix that has the value 1 above the diagonal and 0 elsewhere.

   (a) Find $\exp(tJ_0)$.

   (b) Verify that the identity matrix I and J_0 commute.

   (c) Use this to find $\exp(t(\lambda I + J_0))$.

   (d) Describe the unique solution to the differential equation $y' = Ay$ where A is a matrix in Jordan canonical form.

3. Solve $y' = Ay + g(t)$ with

   (a)
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g(t) = \begin{pmatrix} 2e^t \\ t^2 \end{pmatrix}$$

   (b)
$$A = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \qquad g(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Check Picard iteration on the equation $y' = Ay$.

5. Let y_i be solutions on $[t_0, t_f]$ to $y' = \phi_i(t, y)$ such that $y_i(t_0) = y_i^0$, $i = 1, 2$. Suppose that $||f - g||_\infty < \epsilon$, then for some $K > 0$, show that

$$|y_1(t) - y_2(t)| \leq (|y_1^0 - y_2^0| + \epsilon(t_f - t_0)) \exp(K(t - t_0)).$$