

**INTEGRATION WORKSHOP 2003
COMPLEX ANALYSIS EXERCISES**

DOUGLAS ULMER

1. MEROMORPHIC FUNCTIONS ON THE RIEMANN SPHERE

It's often useful to allow functions to take the value ∞ . This exercise outlines one way to do this for analytic functions.

1.1. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the *extended plane* or the *Riemann sphere*. (Stereographic projection justifies this name.) If $U \subset \mathbb{C}$ is an open set, we say that a function $f : U \rightarrow \hat{\mathbb{C}}$ is *meromorphic* if there exist open subsets U_0 and U_∞ of U such that:

- $U = U_0 \cup U_\infty$
- $f(U_0) \subset \mathbb{C}$ and $f|_{U_0}$ is holomorphic
- $f(U_\infty) \subset \hat{\mathbb{C}} \setminus \{0\}$ and $(1/f)|_{U_\infty}$ is holomorphic

(In the second condition we interpret $1/\infty$ as 0.) Let f be a rational function i.e., a ratio of polynomials. Prove that f defines a meromorphic function $\mathbb{C} \rightarrow \hat{\mathbb{C}}$ in a natural way. Describe carefully the set $f^{-1}(\infty)$.

Prove that $e^{1/z} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ cannot be extended to a meromorphic function $\mathbb{C} \rightarrow \hat{\mathbb{C}}$.

1.2. We also want to define holomorphic and meromorphic functions on $\hat{\mathbb{C}}$. Define a topology on $\hat{\mathbb{C}}$ that is reasonable in light of the previous exercise. (For example, meromorphic functions should be continuous.) For an open set $U \subset \hat{\mathbb{C}}$, a function $f : U \rightarrow \hat{\mathbb{C}}$ is holomorphic (resp. meromorphic) if:

- $f(z)$ restricted to $U \cap \mathbb{C}$ is holomorphic in the usual sense (resp. meromorphic in the sense of the previous exercise)
- $f(1/z)$ restricted to $U \cap (\hat{\mathbb{C}} \setminus \{0\})$ is holomorphic in the usual sense (resp. meromorphic in the sense of the previous exercise)

(Again, we interpret $1/\infty$ as 0.) Prove that a rational function defines a meromorphic function $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Describe carefully $f(\infty)$ and $f^{-1}(\infty)$.

1.3. Let $f = p(z)/q(z)$ where p and q are polynomials with no common factors. Let $d = \max(\deg p, \deg q)$. Show that for all $\lambda \in \hat{\mathbb{C}}$, $f^{-1}(\lambda)$ consists of at most d points, and for all but finitely many λ it consists of exactly d points. What is the maximum number of λ such that $\#f^{-1}(\lambda) < d$?

2. LINEAR FRACTIONAL TRANSFORMATIONS

2.1. Given a 2×2 matrix of complex numbers $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define a meromorphic function $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by

$$z \mapsto S(z) = \frac{az + b}{cz + d}.$$

Prove that:

- S is non-constant if and only if $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. In this case, we say that S is a *linear fractional transformation* (or *Moebius transformation*).
- Composition of LFTs corresponds to multiplication of matrices.
- An LFT S is invertible in the sense that there exists another LFT S' such that $S \circ S' = id$.
- Two LFTs are equal if and only if the corresponding matrices are scalar multiples of one another.

We could summarize this discussion by saying that LFTs form a group, isomorphic to $PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/\mathbb{C}^\times$. This group is sometimes called the *Moebius group*.

2.2. A *circle* in $\hat{\mathbb{C}}$ is either a Euclidean circle $\{z \mid |z - a| = r\} \subset \mathbb{C}$ or a line in \mathbb{C} together with ∞ . The latter are called *circles through ∞* . Prove that an LFT sends circles to circles. Which LFTs send circles through ∞ to circles through ∞ ? Which LFTs preserve the circle $\mathbb{R} \cup \infty$? Which LFTs preserve the circle $|z| = 1$?

2.3. Given three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, prove that there exists a unique LFT S with $S(z_1) = 1$, $S(z_2) = 0$, and $S(z_3) = \infty$. Conclude that the Moebius group acts transitively on the set of triples of distinct points in $\hat{\mathbb{C}}$.

The *cross ratio* of four points z_0, \dots, z_3 is by definition $S(z_0)$ where S is the unique LFT with $S(z_1) = 1$, $S(z_2) = 0$, and $S(z_3) = \infty$. Write down a formula for the cross ratio. Does your formula need any special interpretation when one of the z_i is ∞ ?

2.4. (Possibly rather painful) The cross ratio is not invariant under permutations of the four points. Find an invariant, call it j , which is. Properly constructed, your invariant will have the property that (with the obvious interpretation of the notation), $j(\{z_0, \dots, z_3\}) = j(\{z'_0, \dots, z'_3\})$ if and only if some LFT carries $\{z_0, \dots, z_3\}$ to $\{z'_0, \dots, z'_3\}$.

2.5. Which LFTs preserve the unit disk $D = \{z \mid |z| < 1\}$? Prove that given any $p \in D$, there is an LFT S preserving D such that $S(p) = 0$. Prove that given any circle in $\hat{\mathbb{C}}$ and any point not on the circle, there is an LFT sending the circle to the unit circle $|z| = 1$ and the point to 0.

Without doing too much more calculation, prove similar statements where D is replaced by the upper half plane $\mathbb{H} = \{z \mid \Im z > 0\}$.

3. EXPLICIT CONFORMAL MAPS

The Riemann mapping theorem says that any two connected and simply connected open sets of \mathbb{C} (except \mathbb{C} itself) are conformally equivalent. The proof, however, only gives the existence, not a construction. This exercise builds up a small catalog of conformal maps.

3.1. Find an LFT mapping the upper half plane to the unit disk.

3.2. Find a power function mapping a wedge $\theta_1 < \arg z < \theta_2$ to the upper half plane.

3.3. Find a conformal map from the region $\{|z| < 1\} \cap \{|z - 1| < 1\}$ to a wedge.

3.4. Find a conformal map from an infinite strip $a < \Im z < b$ to the upper half plane

3.5. Find a conformal map from the semi-infinite strip $-\pi/2 < \Re z < \pi/2$, $\Im z > 0$ to the upper half plane.

3.6. Note that one may compose some of these maps and their inverses to build up more complicated conformal maps.

4. MORE ON THE Γ FUNCTION

4.1. A twice differentiable function of a real variable is *convex* if $f''(x) \geq 0$ for all x in its domain. The function f is said to be *log-convex* if its logarithm is convex, or equivalently, if $ff'' - (f')^2 \geq 0$ for all x in the domain. For convenience, we assume below that f is defined on an open interval (a, b) .

4.1.1. Prove that f is convex if and only if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}(f(x_1) + f(x_2))$$

for all $x_1, x_2 \in (a, b)$.

4.1.2. Prove that if f and g are log-convex, then so is $f + g$.

4.1.3. Prove that if $f(z, t)$ is log-convex as a function of z for all t , then a definite integral $F(z) = \int f(z, t) dt$ is log-convex.

4.1.4. Prove that the Γ function restricted to the real axis is log convex.

4.2. We now use log-convexity and the functional equation to give a product expression for $\Gamma(x)$.

4.2.1. Use log-convexity to show that for $x \in (0, 1)$ and $n \geq 2$ an integer, we have

$$\frac{\log \Gamma(-1 + n) - \log \Gamma(n)}{(-1 + n) - n} \leq \frac{\log \Gamma(x + n) - \log \Gamma(n)}{(x + n) - n} \leq \frac{\log \Gamma(1 + n) - \log \Gamma(n)}{(1 + n) - n}.$$

4.2.2. Conclude that

$$(n - 1)^x (n - 1)! \leq \Gamma(x + n) \leq n^x (n - 1)!$$

4.2.3. Use the functional equation to deduce that

$$\frac{n^x n!}{x(x + 1) \cdots (x + n)} \leq \Gamma(x) \leq \frac{n^x n!}{x(x + 1) \cdots (x + n)} \frac{x + n}{n}$$

and so

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n^x n!}{x(x + 1) \cdots (x + n)}$$

4.2.4. Now show that the limit on the right hand side of the last displayed equation exists for all $x \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and that the resulting function satisfies the same functional equation as Γ and so is equal to Γ on that domain. (Let $\Gamma_n(x)$ be the expression under the limit sign and consider $\Gamma_n(x + 1)$ to deduce a functional equation which tends to that of Γ .)

4.2.5. Show that

$$\Gamma_n(x) = e^{x(\log n - 1 - 1/2 - \dots - 1/n)} \frac{1}{x} \frac{e^x}{1+x} \frac{e^{x/2}}{1+x/2} \dots \frac{e^{x/n}}{1+x/n}$$

and that the limit

$$\gamma = \lim_{n \rightarrow \infty} \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right)$$

exists. It is called *Euler's constant* and its value is about .57722. Conclude that

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n}.$$

4.3. It is a theorem that Γ is characterized by the properties: Γ is log-convex on $(0, \infty)$, $\Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$. There is lots more interesting to say about the Γ function both from the classical point of view and via algebraic geometry. See Ahlfors's or Artin ("The Gamma Function") for the classical story and the work of Greg Anderson and Yasutaka Ihara for some modern ideas.

5. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

5.1. Use the Cauchy integral formula to prove that if f is analytic on \mathbb{C} and $|f|$ is bounded then f is a constant. This is Liouville's theorem.

5.2. Deduce the fundamental theorem of algebra: if f is a polynomial with no roots in \mathbb{C} , then f is a constant.

5.3. For another proof of FTA, recall from above that a polynomial gives a continuous map $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ which sends ∞ to ∞ . Prove that if the extended map is not constant then it is open and deduce that it must be surjective.

6. MEROMORPHIC FUNCTIONS ON THE RIEMANN SPHERE

6.1. Generalize Liouville's theorem by showing that if f is analytic on \mathbb{C} and $|f(z)| < |z^n|$ for some n and all sufficiently large $|z|$, then f is a polynomial.

6.2. Prove that a meromorphic function $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is in fact a rational function. (Hint: Partial fractions! Show that f has finitely many poles in the finite plane and for suitable constants $a_{n,p}$, $f - \sum_p \sum_{n < 0} a_{n,p} (z-p)^n$ is holomorphic in the finite plane and has at worst a pole at ∞ . Now apply the previous exercise.)

6.3. Deduce that the invertible meromorphic functions $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ are precisely the LFTs. I.e., $\text{Aut}(\hat{\mathbb{C}}) = PGL_2(\mathbb{C})$.

7. HARMONIC FUNCTIONS

7.1. Prove that the real and imaginary parts of an analytic function are *harmonic*, i.e., they satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

7.2. Given a harmonic function u on a connected open subset $U \subset \mathbb{C}$ and $p \in U$, show that there is a harmonic function v defined on a neighborhood $V \subset U$ of p such that $u + iv$ is analytic on V . Although it's not unique, v is called the harmonic conjugate of u .

Show by example that we cannot insist that $V = U$.

7.3. Show that if u is harmonic, then $f = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$ is analytic.

7.4. Use the Cauchy integral formula and 7.2 to show that if u is harmonic on an open set U , $z \in U$ and $r > 0$ is a sufficiently small real number, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta.$$

7.5. Use an LFT and the previous exercise to show that if u is harmonic on the disk $|z| < R$ and $|a| < R$ then

$$\begin{aligned} u(a) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |a|^2}{|Re^{i\theta} - a|^2} u(Re^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \Re \left(\frac{Re^{i\theta} + a}{Re^{i\theta} - a} \right) u(Re^{i\theta}) d\theta. \end{aligned}$$

This is the Poisson integral formula giving the values of a harmonic function on a disk in term of its boundary values.

8. DIRICHLET PROBLEMS

The *Dirichlet problem* is to find a harmonic function on a domain with specified boundary values. The Poisson integral formula of the previous exercise solves this problem for a disk. More precisely, if a function u is given on the unit circle, then the formula defines a harmonic function on the interior of the disk. It is a theorem of H. A. Schwarz that if u is continuous at $p \in S^1$, then the function defined by the integral tends to $u(p)$ as $z \rightarrow p$. Using conformal mapping one may solve the Dirichlet problem in other domains.

8.1. Find a harmonic function on the unit disk with boundary values 1 on $\{z \mid |z| = 1, \Im z > 0\}$ and 0 on $\{z \mid |z| = 1, \Im z < 0\}$. Hint: Use the branch of arctan with $0 \leq \arctan t \leq \pi$.

Find a harmonic function on the upper half plane with boundary values 0 on $\{x \mid |x| > 1\}$ and 1 on $\{x \mid |x| < 1\}$.

One may interpret this function in terms of, say, steady state temperatures in a thin plate. The harmonic conjugate then may be interpreted in terms of heat flow.

8.2. See the book of Churchill and Brown for many more applications of this type.

9. DEFINITE INTEGRALS VIA THE RESIDUE CALCULUS

It's possible to evaluate some real definite integrals using the residue calculus.

9.1. Since $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, an integral of the form

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta \quad *$$

where R is a rational function can be rewritten

$$\int_{|z|=1} S(z) \frac{dz}{iz}$$

where S is again a rational function. The latter is easily evaluated via the residue calculus. Use this method to evaluate

$$\int_0^{2\pi} \frac{\sin^2 \theta}{\cos \theta + 1} d\theta.$$

Note that this is *a priori* an improper integral. (In this case, you may check your work by evaluating the integral using the methods of freshman calculus. Integrals of the form $*$ can always be evaluated by freshman calculus methods, but the residue method is sometimes more efficient.)

9.2. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$$

by relating it to $\lim_{R \rightarrow \infty} \int_{C_R} \frac{dz}{z^2 + 1}$ where C_R is the closed path traversing $[-R, R]$ followed by a semicircle of radius R .

9.3. Use a similar idea to evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 + 1} = \Im \int_{-\infty}^{\infty} \frac{ze^{iz} dz}{z^2 + 1}$$

using a limit of rectangular contours. Be careful to justify the convergence.