INTRODUCTION TO COHERENT STATES: CONVERGENCE ISSUES AND EXISTENCE OF INTEGRABLE P-REPRESENTATIONS

1. Introduction

The concept of coherent states and their applications can be traced to early literature in the field of quantum optics and was widened for more general applications, primarily in quantum mechanics. In this project we introduce the ideas vital to understanding the definition of coherent states and investigate some of the most basic results about these objects, in particular, the so-called P-representation of certain linear (not neccesarily bounded) operators on the Hilbert space $L^2(\mathbf{R}, \mathbf{dx})$. Lastly, we briefly describe the connection between coherent states and the quantumness of states of the harmonic oscillator.

2. Preliminaries

2.1. **Operator Convergence.** Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbf{R}, \mathbf{dx})$ with inner product given by

$$(f, g) = \int_{\mathbf{R}} \overline{f(x)} g(x) dx \tag{1}$$

Also let $\mathcal{L}(\mathcal{H})$ denote the space of bounded linear operators from $\mathcal{H}$ to itself, a Banach Space. Then given an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$, for all y $\in \mathcal{H}$ we may write,

$$y = \sum_{n=1}^{\infty} (\phi_n, y)\phi_n \tag{2}$$

The previous equation is true in the sense that the partial sums of the right hand side converge, in the Hilbert space norm, to y. We state the following classical theorem so we can use it to clarify definitions of several topologies on $\mathcal{L}(\mathcal{H})$ that follow.

Theorem (The Hilbert-Schmidt Theorem)**.** *If A is a compact, self-adjoint operator on H, then there exists $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbf{C}$, $\lambda_n \to 0$ and an orthonormal basis of eigenvectors ϕ_n, corresponding to eigenvalues λ_n, so that,*

$$Ay = \sum_{n=1}^{\infty} \lambda_n (\phi_n, y)\phi_n \tag{3}$$

As with (2), equation (3) is true in the sense that the partial sums of the right hand side converge. We would like to abreviate our notation by stating (3) as follows:

$$A = \sum_{n=1}^{\infty} \lambda_n (\phi_n, \cdot)\phi_n \tag{4}$$

This statement only makes sense if one identifies the topological sense of convergence that is intended. We will see that our statement of the Hilbert-Schmidt theorem is equivalent to saying that (4) holds in the strong operator topology sense given the antecedent specified in the theorem. The following definitions are in line. First recall,

Definition. *For a bounded operator $A \in \mathcal{L}(\mathcal{H})$, the* **operator norm** *of A is given by*

$$\|A\| := \sup_{\|x\|=1} \|Ax\| < \infty \tag{5}$$

Definition. *The topology on $\mathcal{L}(\mathcal{H})$ induced by the operator norm is called the* **uniform operator topology.**

A topology different from the uniform operator topology is defined as follows,

Definition. *The* **strong operator topology** *is the weakest topology on $\mathcal{L}(\mathcal{H})$ such that E_y:$\mathcal{L}(\mathcal{H}) \to \mathcal{H}$, given by $A \mapsto Ay$, is continuous for all $y \in \mathcal{H}$.*

Because of our choice of terminology it sounds silly, but it is in fact true, that the strong operator topology is a weaker topology than the uniform operator topology. Clearly equation (3) of the Hilbert-Schmidt theorem just says that equation (4) holds in the strong operator topology sense. In fact, we can say more: Since,

$$\|A - \sum_{n=1}^{N} \lambda_n(\phi_n, \cdot)\phi_n\| = \sup_{\|x\|=1} (\|Ax - \sum_{n=1}^{N} \lambda_n(\phi_n, x)\phi_n\| \to 0, \quad \text{as} \quad N \to \infty, \tag{6}$$

we may make the stronger statement that (4) holds in the uniform operator topology sense (In general, uniform operator topology convergence implies strong operator topology convergence). Lastly we define the weak operator topology on $\mathcal{L}(\mathcal{H})$.

Definition. *The* **weak operator topology.** *is the weakest topology on $\mathcal{L}(\mathcal{H})$ such that $E_{y,f}$:$\mathcal{L}(\mathcal{H}) \to \mathbf{C}$, given by $A \mapsto (f,Ay)$, is continuous for all $y,f \in \mathcal{H}$.*

Convergence in the strong operator sense implies convergence in the weak operator sense so it follows from the Hilbert-Schmidt Theorem that equation (4) holds in the weak operator topology sense. To comprehend the existence of the coherent states that we will define, and their applications, it is essential to understand the senses of convergence for which the formal operator equalities we will write down hold. Commonly, we will resign to convergence the weak operator topology sense.

2.2. **Hermite Functions.**

Definition. *Let $f \in L^2(\mathbf{R}, \mathbf{dx})$. Then we say f is in* **Schwarz Space***, denoted $f \in \mathcal{S}(\mathbf{R})$, if f is infinitely differentialbe and for all n,$m \in \mathbf{N}$,*

$$\sup_{x\in\mathbf{R}} |x^n \frac{d^m}{dx^m}\phi(x)| < \infty \tag{7}$$

Definition. *We define a,$a^\dagger$,N: $\mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$ as follows: For $\phi \in \mathcal{S}(\mathbf{R})$,*

$$a\phi := (x + \frac{d}{dx})\phi \quad a^\dagger\phi := (x - \frac{d}{dx})\phi \quad N\phi := a^\dagger a\phi \tag{8}$$

Note that we also have,

$$N = \frac{1}{2}(H - I) \text{ where } H \text{ is the harmonic oscillator, } H(\cdot) := (-\frac{d^2}{dx^2} + x^2)(\cdot) \tag{9}$$

and

$$[a, a^\dagger] := aa^\dagger - a^\dagger a = I \tag{10}$$

It can be shown that $\phi_0 := \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}$ and $\phi_n := (n!)^{-\frac{1}{2}}(a^\dagger)^n \phi_0$ are the eigenfunctions of H; that is, $H\phi_n = (2n+1)\phi_n$ for all n$\in\mathbf{N}$. It is a nontrivial fact that $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. It then follows from equation (2) and the preceeding definition that $I = \sum_{n=1}^{\infty}(\phi_n, \cdot)\phi_n$ in the strong operator topology sense. The following theorem will also be of use to us.

Theorem. *If $A, B: \mathcal{H} \to \mathcal{H}$ are linear operators that both commute with $[A, B]$, then*

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \tag{11}$$

We are now ready to introduce coherent states.

3. Coherent States and Resolution of Unity

3.1. Canonical Coherent States.

Definition. *Let $z \in \mathbf{C}$ and let $z^* \in \mathbf{C}$ denote the conjugate of z. The* **cannoncial coherent states** *are given by*

$$\phi_z(x) := e^{(za^\dagger - z^* a)} \phi_0(x) \tag{12}$$

It follows that

$$\begin{aligned} \phi_z(x) :&= e^{-\frac{1}{2}[za^\dagger, -z^* a]} e^{za^\dagger} e^{-z^* a} \phi_0(x) \\ &= e^{-\frac{1}{2}|z|^2} e^{za^\dagger} (I - z^* a + \frac{(z^* a)^2}{2!} - \dots) \phi_0(x) \\ &= e^{-\frac{1}{2}|z|^2} e^{za^\dagger} \phi_0(x) \\ &= e^{-\frac{1}{2}|z|^2} (I + za^\dagger + \frac{(za^\dagger)^2}{2!} + \dots) \phi_0(x) \\ &= e^{-\frac{1}{2}|z|^2} \sum_{n=1}^{\infty} \frac{1}{n!} z^n \phi_n(x) \end{aligned} \tag{13}$$

Thus for $z_1, z_2 \in \mathbf{C}$, we also have,

$$(\phi_{z_2}, \phi_{z_1}) = e^{-\frac{1}{2}|z_2|^2 + z_2^* z_1 - \frac{1}{2}|z_1|^2} \tag{14}$$

3.2. **Resolution of Unity.** In the following we use the notation $d^2z := d(Re(z))d(Im(z))$. Let $\phi, \psi \in \mathcal{H}$. Then the function $z \mapsto (\phi, \phi_z)(\phi_z, psi)$ is integrable, as follows:

(15)

$$\begin{aligned}\int_{\mathbf{R}^2}(\phi,\phi_z)(\phi_z,\psi)d^2z &= \int_{\mathbf{R}^2}[e^{-\frac{1}{2}|z|^2}\sum_{n=1}^{\infty}\frac{1}{n!}z^n(\phi,\phi_n(x))][e^{-\frac{1}{2}|z|^2}\sum_{m=1}^{\infty}\frac{1}{m!}(z^*)^m(\phi_m(x),\psi)]d^2z\\ &= \int_{\mathbf{R}^2}\sum_{n,m=1}^{\infty}\frac{e^{-|z|^2}}{\sqrt{n!m!}}z^n(z^*)^m(\phi,\phi_n)(\phi_m,\psi)d^2z\\ &= \sum_{n,m=1}^{\infty}\frac{1}{\sqrt{n!m!}}\int_{\mathbf{R}^2}e^{-z^2}|z|^ne^{in\theta}|z|^me^{-im\theta}(\phi,\phi_n)(\phi_m,\psi)|z|d\theta d|z|\\ &= \sum_{n=1}^{\infty}\frac{2\pi}{n!}\int_{\mathbf{R}}e^{-|z|^2}|z|^{2n}(\phi,\phi_n)(\phi_n,\psi)|z|d|z|\\ &= \sum_{n=1}^{\infty}\frac{\pi}{n!}\int_{\mathbf{R}}e^{-|z|^2}|z|^{2n}(\phi,\phi_n)(\phi_n,\psi)d|z|^2\\ &= \pi\ \sum_{n=1}^{\infty}(\phi,\phi_n)(\phi_n,\psi)\\ &= \pi\ (\phi,\psi) = \pi\ (\phi, I\psi)\end{aligned}$$

Or, equaivalently, $I = \frac{1}{\pi}\int(\phi_z,\cdot)\phi_z d^2z$ in the weak operator topology sense.

Remark. In physics, one often writes $I = \frac{1}{\pi}\int |z><z|d^2z$ for the previous expression. This is, of course, an equivalent form of the expression writtem in Dirac's bra-ket notation. One associates the 'bra' $<z|$ with the functional $(\phi_z,\cdot)$ and associates the 'ket' $|\phi>$ with the the function ϕ_z.

An interestesting question that was considered but not resolved in our project concerns whether the previous identity true in the strong operator topology sense. That is, does$I\psi = \frac{1}{\pi}\int(\phi_z,\psi)\phi_z d^2z$ for all $\phi \in \mathcal{H}$? To answer this question, one would presumably have to consider the right hand side of the equation in question as a Bochner integral: an integral of a Hilbert Space valued function over **C**.

4. Diagonalization of Arbitrary Linear Operators With Respect to Overcomplete Basis of Coherent States

We will use the following defintions and notations in this section, further description of these spaces can be found in [Gelfand]:

$S'(\mathbf{R^2})$ is the dual of $\mathbf{S}(\mathbf{R^2})$, namely the **tempered distributions** over $\mathbf{R^2}$.

$D(\mathbf{R^2}) := \{\mathbf{f} \in \mathbf{L^2}(\mathbf{R},\mathbf{dx}) :\ \mathbf{f}$ is infinitely differentiable over $\mathbf{R^2}\}$.

$D'(\mathbf{r^2})$ is the dual of $\mathbf{D}(\mathbf{R^2})$, namely the **distributions** over $\mathbf{R^2}$.

$Z(\mathbf{R^2}) :=$
$\{f \in L^2(\mathbf{R}, \mathbf{dx}) :$ $\mathbf{f}$ is entire over $\mathbf{C^2}$ and $\forall \mathbf{n}, \mathbf{m} \in \mathbf{N}$, $\exists \mathbf{C_{nm}}, \mathbf{a}, \mathbf{b} \geq \mathbf{0}$,
such that$|x^n y^m f(x,y)| \leq C_{nm} e^{a|\Im(x)| + b\Re(x)}\}$.

$Z'(\mathbf{r^2})$is the dual of$\mathbf{Z}(\mathbf{R^2})$, namely the **ultradistributions** over $\mathbf{R^2}$.

Based on our results from the previous section, we will borrow jargon from linear algebra, and say that the function $f(z) := \frac{1}{\pi}$ diagonalizes the identity operator on $\mathcal{H}$ with respect to the 'overcomplete' (i.e. linearly dependent) basis of coherent states. This langague suggests that there are other operators in $\mathcal{L}(\mathcal{H})$ that can be diagonalized by some function $f : \mathbf{C} \to \mathbf{C}$ with respect to the coherent states, ϕ_z. This is true in some cases, but we will need to allow the diagonalization function to belong to a broader range of objects, namely distributions, and we will also find that there is class of unbounded linear operators on (H) that can be diagonalized in such a way. We further motivate this idea with the following observations. Let $A \in \mathcal{L}(\mathcal{H})$ or be a polynomial in the operators a and $a^\dagger$ (say, $A = \sum_{n,m=1}^{2N,2M} c_{jk} (a^\dagger)^j a^k$ for some $c_{j,k} \in \mathbf{C}$). It follows from equation (14) that, for $z_1, z_2 \in \mathbf{C}$, we have,

$$(\phi_{z_1}, A\phi_{z_2}) = e^{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2} \sum_{n,m=1}^{\infty} \frac{1}{\sqrt{n!m!}} (\phi_n, A\phi_m)(z_1^*)^n (z_2)^m \tag{16}$$

Two things to note here are (i) the above series defines and entire function in z and z^*, and (ii) such a function is determined by its values on any uncountable subset of the complex plane, in particular, the diagonal $z_1 = z_2$. Thus A is completely determined by the 'diagonal elements' $(\phi_z, A\phi_z)$. With these facts as our motivation, we make the following conjecture (stated as a theorem in [Miller, Mishkin]).

Theorem. *Every linear operator $A : \mathcal{H} \to \mathcal{H}$ that is either bounded or polynomial in a and $a^\dagger$ has a P-representation,*

$$(\phi, A\psi) = \int P(z)(\phi, \phi_z)(\phi_z, \psi) d^2 z \tag{17}$$

where $P(z)$ is an ultadistribution on $\mathbf{R^2}$

Following is a very rough sketch of the proof of this theorem given by Miller and Mishkin; we do not claim this proof is correct or incorrect. The proof first shows that the function $A : \mathbf{R^2} \to \mathbf{C}$ given by $A(x,y) := (\phi_{x+iy}, A\phi_{x+iy})$ is a tempered distribution, therefore an ultradistribution, on $\mathbf{R^2}$. Also, if $P(x,y) := P(x+iy)$ is a P-representation for A, then it must be that $A(x,y) = P(x,y) * e^{-(x^2+y^2)}$. Taking the Fourier transform of each side of this equation and noting that the Fourier transform maps ultradistributions to distributions, gives,

$$\tilde{A}(u,v) = \pi \tilde{P} e^{\frac{-(u^2+v^2)}{4}} \subseteq D'(\mathbf{R^2}) \tag{18}$$

Since the function $e^{\frac{u^2+v^2}{4}}$ is a multiplier in $D(\mathbf{R^2})$ (that is $e^{\frac{u^2+v^2}{4}} \cdot f \in D(\mathbf{r^2})$ whenever f is), it follows that, $\tilde{P}$ is also a distibution. Taking the inverse Fourier transform then yields the desired P-representation, (17). According to the theorem, we then have that,

$$A = \int P(z)(\phi_z, \cdot)\phi_z d^2z \text{ in the weak operator topology sense,} \tag{19}$$

where $P(z)$ is an ultradistribution.
However, this would inply that one needs the map $z \mapsto (\phi, \phi_z)(\phi_z, \psi)$ to be in $Z(\mathbf{R^2})$ for all $\phi, \psi \in \mathcal{H}$ for the equality (17) to be meaningful; this is a statement we were unable to verify and seems to be, in general, untrue.

5. Ambiguities and Applications

In conclusion, for some $A \in \mathcal{L}(\mathcal{H})$ or polynomial in a and $a^\dagger$, the P-representation, $P(z)(\phi, \phi_z)(\phi_z, \psi)$ is not in fact integrable. However, there are states of the harmonic oscillator for which this P-representation is integrable and useful in applications, as can be seen in several of the publications listed below. In particular, if a state has a positive, integrable P-representation, then the quantum statistical properties of the oscillator in that state are the same as those of a classical statistical mechanics ensemble: They are described by means of the measure μ defined by,

$$\mu(\Omega) = \int_\Omega P(z) d^2z \geq 0 \tag{20}$$

Thus we could perhaps talk about the extent of quantumness of states of the harmonic oscillator [Korbicz,Cirac,Wehr].

References

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