VIGRE Report for Fall 2002

Jeff Selden

January 14, 2003

1 Progress Towards Degree

Let us begin with a short description of the questions on which I have been spending my time. Consider a graph G in $\mathbb{R}^2$, i.e., the standard type of a graph with edges and vertices. Now we will fatten the graph so that it has a thickness at each point proportional to ϵ. We do not require that the thickness be constant. This process gives us a domain that we will call our fattened graph G_ϵ. Since G_ϵ is a domain in $\mathbb{R}^2$, we can consider the laplacian Δ with Dirichlet boundary conditions (one could consider Neumann boundary conditions as well). We are interested in the asymptotic behavior of the eigenvalues of Δ as ϵ tends to 0.

Much of the existing results in this area have already been discussed in previous VIGRE reports (see Fall 2001, Summer 2002), so we will only briefly mention what is known. The 0th order asymptotic behavior of the eigenvalues is known, i.e., we get the eigenvalues of a second-order ODE on the graph with some Kirchhoff-type laws at the vertices. Therefore, our goal is to understand some of the higher order asymptotics. To do so, we have simplified the problem from a graph to a line segment. So the setup is that we have a positive function $f(x)$ on an interval (a, b) and we are considering the domain bounded by the x-axis, the vertical lines $x = a$ and $x = b$, and the function $y = \epsilon f(x)$.

One approach that was introduced in the Fall 2001 VIGRE report is the Birman-Schwinger technique which is particularly well-suited for perturbative eigenvalue problems. It essentially swaps the roles of the perturbation and spectral parameters, thereby provided the researcher with a new eigenvalue problem which may (and hopefully is) a little easier to handle. Unfortunately for us, our problem is singular and this seems to prevent the technique from working its magic. In particular, it appears that one needs to have an isolated eigenvalue at the bottom of the spectrum in order for the Birman-Schwinger technique to apply and the bottom of our spectrum is continuous (the unperturbed operator is the composition of two orthogonal operators, one of which is a multiplication operator). On the bright side, I was able to use the technique to obtain results about the asymptotics for the eigenvalues of a related (yet easier) problem. If we bound the domain by $y = 1 + \epsilon f(x)$, we have a regular perturbation problem and Birman-Schwinger works. Unfortunately, these results were already well-known and present in the literature.

The latest attempt on the problem takes a somewhat indirect approach. There are results in the literature dealing with what one could call the "exterior" problem, i.e., the behavior of the eigenvalues of Δ on a domain with obstacles as the size of the obstacles goes to 0. For example, there are results for a domain with a narrow slit obstacle as the width of the slit goes to 0. The question then becomes whether or not one can connect the exterior problem with our problem in a way that allows us to use their asymptotics. A possible answer may come from the Dirichlet-to-Neumann operator. This operator connects the eigenvalue problems for the laplacian on domains with common boundaries. My investigations into this approach are still at the initial stages, but it is at least promising.

2 Fall 2002 Courses Taken

Though I was not officially registered for the course, I attended Prof. Friedlander's lectures on functional analysis.

3 Fall 2001 Seminars Attended

- Graduate Colloquium
- Geometry Seminar
- Analysis and Its Applications Seminar
- Mathematics Colloquium