Print Form

DEPARTMENT OF MATHEMATICS

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year: Fall/2008

Name: Daniel Champion

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor
PHYS 569	Introduction to General Relativity	A	Dimitrios Psaltis
MATH 920	Dissertation	S	David Glickenstein

List the title, date and location of any talks you have given, either here or elsewhere:

"Circle Trios and Sphere Quartets," 11-18-09, Geometry Seminar.

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

I made progress researching known results in the field of variational techniques on two and three dimensional simplicial manifolds. I was able to fully extend the known two dimensional circle trio results to all dimensions, and I extended the results to hyperideal and ideal circle trios and sphere quartets. In this same category, I explicitly parametrized classes of sphere quartets obtaining an explicit description of the configurations. in these classes.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- [ ] Independent Study
- [ ] Oral Comprehensive Exam
- [x] Commence Thesis Research
- [ ] Conference attendance
- [ ] Conference participation
- [ ] Complete PhD

Professional development and outreach:

- [ ] AP Calculus Visit
- [ ] High School Workshops
- [ ] Undergraduate Research Project
- [ ] Undergraduate Research Seminar
- [ ] Super TA
- [ ] Mentoring junior graduate students for the qualifying exams
- [ ] RTG (help organize)
- [ ] Research Seminar (help organize)

- [x] Other (please specify)

I worked as a mentor and advisor for undergraduates working on Prof. Glickenstein's combinatorial geometry laboratory.

Attach a brief statment about your academic progress and professional development during the period of support.

VIGRE Report, Fall 2008

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1 Academic Progress

1.1 Course work

During the fall 2008 period of VIGRE support my academic progress was primarily research for my dissertation. However I also took the "Introduction to General Relativity" course PHYS 569. This course is a natural course for any student of differential geometry and it satisfied half of my out-of-department course work requirements for the Ph.D. degree. I received an A in the class, and wrote a term paper on multiply connected structures of the universe and the relation to the cosmic microwave background radiation anisotropies.

1.2 Dissertation Research

Under advisement by Prof. David Glickenstein, I made progress researching and solving several problems related to placing global metric structure on simplicial manifolds. Simplicial manifolds are n-dimensional simplicial complexes (usually $n = 2$ or 3). In the case when $n = 2$, each 2-simplex can be assigned angles for each of its corners, or each 1-simplex (edge) can be assigned a *dihedral angle*. Given either of these assignments, a natural question is:

Problem 1 *When can the triangles be scaled relative to the corner angle or dihedral angle assignment to produce a global (possible immersed) gluing of all the edges in a consistent manner.*

The corresponding problem for $n = 3$ is:

Problem 2 *When can the simplices of a simplicial* $3-$*manifold be scaled relative to an assignment of a suitable face invariant to produce a global gluing of all the codimension-1 faces in a consistent manner.*

For example, if each corner of each triangle of a simplicial surface is assigned the angle $\frac{\pi}{3}$, we can scale every (equilateral) triangle to have edges of length 1, producing a trivial global gluing. However, it is easy to produce angle assignments that do not have global gluings.

Given a simplicial surface S, we can produce a hyperbolic ideal polyhedron Q (combinatorial) corresponding to S in the following way:

1. For any triangle t of S, place t on the sphere at infinity of the upper half space model of hyperbolic 3-space.
2. Let A_t be the ideal tetrahedron with the vertices of t plus the point $\{\infty\}$.
3. Let Q be the 3-dimensional simplicial complex defined by the tetrahedra A_t and the combinatorics of S.

We define the *volume* of Q to be the sum of the volumes of all the tetrahedra A_t for $t \in S$. In this setting, we can actualize the dihedral angle of an edge e common to the triangles t and t' of S as the dihedral angle of the hyperbolic ideal polyhedron produced by scaling and translating A_t and $A_{t'}$ so that they meet along the edges corresponding to e. When dihedral angles are assigned to the edges, the surprising answer to Problem 1 is that the hyperbolic volume of Q is maximized when a global gluing exists. From this result, there are many "dimensions of generalization" possible where I have worked on similar problems as the one above

1. Non-Euclidean simplicial surfaces (hyperbolic or spherical triangular faces)
2. Weighted simplicial surfaces (hyper-ideal triangular faces), where weights are assigned to each vertex.
3. Circle (sphere) packings on simplicial surfaces; weighted simplicial surfaces such that the sum of the weights of any two adjacent vertices equals the length of the corresponding edge.
4. Use of functionals other than volume.
5. The $n = 3$ problem (very little is known here).
6. Conditions on the dihedral angles that produce a convex functional.

For the preparation of my comprehensive exam I studied extensively the work of Igor Rivin on this subject. Rivin published a foundational paper [Riv] in 1994 "Euclidean Structures on Simplicial Surfaces and Hyperbolic Volume," in which the question mentioned above was answered in the case of Euclidean simplicial surfaces. Specifically, the main result of this paper was:

Theorem 3 *(Rivin) The simplicial surface with prescribed dihedral angles of maximal volume (in the sense described above) is the unique simplicial surface with a global gluing having the prescribed dihedral angles.*

The study of Rivin's work led to the consideration of spherical and hyperbolic simplicial surfaces, of which similar theorems have been proven by Feng Luo in [Luo1] using a functional related to volume. Furthermore, additional complexity can be introduced by assigning weights to the vertices of the simplicial surface. These weights can be visualized as specifying radii of circles centered at each vertex. The global gluing of a weighted simplicial surface would then include the condition that the weights are consistently assigned for every triangle meeting at a given vertex. Theorems similar to Rivin's were proven by Boris Springborn in [Spr] for certain weighted simplicial surfaces. A special case of a weighted simplicial surface is a circle packing on a simplicial surface, in which the weights at the vertices of a given triangle with vertices $\{1, 2, 3\}$ are R_1, R_2, R_3 respectively and satisfy $l_{12} = R_1 + R_2$, $l_{13} = R_1 + R_3$, $l_{23} = R_2 + R_3$ where l_{ij} is the length of the edge $\{i, j\}$. Both Luo and Springborn used functionals that are related to volume, but distinctly different. However, the variation approach of Luo and Springborn was identical to Rivin's (the functional used in each case is shown to be concave down, the maximum is shown to be a global gluing usually by a Lagrange multiplier argument).

During VIGRE support for fall 2008 I was able to make progress on several generalizations of the problems mentioned above; namely I have focused on the 3−dimensional case. The main obstructions in the three dimensional case is the convexity of the volume functional (as applied in a variational argument). Rivin's restriction that the dihedral angles be no more than π ensures convexity of the volume functional. However an equivalent restriction on solid angles for the 3-dimensional case is not clear. However, after research done during Fall 2008 it has become clear that if we consider total scalar curvature and edge lengths instead of volume and dihedral angles the correct restriction can be found to arrange for convexity. These restrictions could be translated to the solid angles via a Legandre transform.

Under VIGRE support for fall 2008 we have shown that many of the 2-dimensional results can be combined into a parsimonious theory. Many papers have been published in this category, and during the last 1.5 years we have made progress developing a meta-argument for the results in this category.

Currently there are obstructions in the theory of weighted simplicial surfaces involving the "amount of intersection" that the weight circles are allowed to have. I have been working on removing these restrictions, which has resulted in my most significant accomplishments during VIGRE support. I will devote the remainder of this section describing these accomplishments.

In the study of simplicial surfaces where the weight circles are allowed to intersect at arbitrary angles, and even further, I have been lead to the study of circle and sphere configurations and the resulting curvilinear polygons and polyhedra they produce. If we consider for a moment two circles in the plane, and study the effects of changing the radius of one of the circles. It is easy to see that the radius of the unrestricted circle can be chosen to:

1. Miss the other circle (empty intersection of the two circles; this corresponds to the work of Springborn on weighted simplicial surfaces),

2. Intersect the other circle at some angle, or

3. Swallow the other circle inside of the unrestricted circle.

Currently only weighted simplicial surfaces where the weight circles pair-wise behave like type 1 have been shown to have rigidity results. In order to extend these results to types 2 and 3 it became necessary to extensively study these configurations.

For a point x, circle/sphere C and ray r from x intersecting ∂C at A and B, the circle/sphere power $P(x,C)$ of x relative to C is defined as:

$$P(x,C) = \begin{cases} -|xA|\,|xB| & \text{if } x \in C, \\ |xA|\,|xB| & \text{if } x \notin C. \end{cases},$$

which can be shown to be independent of the line l. Given two arbitrary circles/spheres, we can construct the equal power line for these circles/spheres, such that any point on this line has the same power with respect to either circle/sphere. Given a trio of appropriately pair-wise intersecting circles, there is beautiful result found in [Del]: the location of the point of intersection of the three equal power lines (the three equal power lines always meet at a common point) determines the geometry of the curvilinear triangles produced in the circle trio. The results of [Del] require that the circle pair-wise intersect at two distinct points. During fall 2008 VIGRE support I was able to remove all conditions on the intersections of the circles in the circle trio as described in the following theorem.

Theorem 4 *Let P be the common power point of three circles in the plane, where $P = \infty$ if the three common power lines do not intersect.*

1. *If P lies within all three circles, then any triangular region formed is spherical.*

2. *If P lies on all three circles, then any triangular region formed that doesn't contain P is Euclidean (possibly degenerate), and if a triangular region contains P then it is a non-compact projective Euclidean triangle.*

3. *If P lies outside all three circles, then any triangular region formed is hyperbolic. Furthermore, any such configuration of circles forms a hyperbolic triangle of possibly mixed type according to the following:*

   - *If circles C_i and C_j intersect at two points, then the vertex of the triangular region corresponding to C_i and C_j is finite.*
   - *If circles C_i and C_j intersect at one point, then the vertex of the triangular region corresponding to C_i and C_j is ideal (infinite).*
   - *If circles C_i and C_j are disjoint, then the vertex of the triangular region corresponding to C_i and C_j is hyperideal (infinite).*

My interests on three dimensional simplicial manifolds led me to study sphere quartets, the natural generalization of circle trios. I was able to generalize the theorem for sphere quartets (indeed for collections of $n + 1$ many n-spheres). Furthermore, during my study of sphere quartets I used the results of [Luo2] to obtain an explicit parametrization of some configurations of sphere quartets which enables the location of tetrahedral regions and the study of degeneracies. The study of the sphere quartets is the first step in generalizing the rigidity results studied by Springborn to three dimensions.

1.3 Talks

I gave a talk in the geometry seminar on Nov. 18 2008 on some of my recent circle trio and sphere quartet results. Furthermore I attend the Geometry seminar, Mathematics colloquium, Analysis and its Applications seminar, and the Graduate Colloquium regularly.

2 Professional Development

As part of my VIGRE funding for fall 2008 I was involved in a broad research project supervised by Prof. Glickenstein involving other graduate students, and undergraduates. The primary objective of this research group is the production of computer models and visualization of combinatorial Ricci flow, Yamabe flow, combinatorial algorithms on simplicial manifolds, and research on other flows. The project began in the summer of 2008 where graduate students and undergraduates worked together on programming and theory. I was involved in the project from the beginning, however as part of VIGRE funding I worked with the undergraduates as a mentor, advisor, and coworker on a daily basis during the fall of 2008. During this time we made progress on all components of the project; some highlights of our accomplishments follow.

The fundamental components of the project include the implementation of combinatorial Ricci flow and Yamabe flow. These flows perturb the metric structure of two or three dimensional simplicial manifolds by using discrete quantities (curvature, variations, etc.) to numerically solve the differential equations governing the Ricci and Yamabe flows. The implementation of both of the flows was done during the summer and fall of 2008.

The Ricci and Yamabe flows are not limited to Euclidean simplices, so a natural extension was to implement the flows for hyperbolic and spherical geometries. The spherical combinatorial Ricci flow was a large part of my work in the project during fall 2008. In particular, I worked on a multitude of theoretical challenges in developing a normalization of spherical combinatorial Ricci flow.

Another large accomplishment during the fall was the construction of a GUI program the efficiently runs and outputs data from either of the two flows. Given the complexity of some triangulations of the simplicial manifolds, such an interface is an invaluable tool for experimentation and data analysis.

The most recent accomplishments in the project include the addition of conformal structure in the Yamabe flow and implementation of a min-max program for the Yamabe flow designed to find positive constant curvature metrics on 3-manifolds.

3 References

References

[Del] C. Delman and G. Galperin, A Tale of Three Circles, Mathematics Magazine, Vol. 76, No. 1 (Feb., 2003), pp. 15-32.

[Luo1] Luo, Feng, A characterization of spherical polyhedron surfaces, J. Differential Geom.**74**, (2006), no. 3, 407-424.

[Luo2] Luo, Feng, On a Problem of Fenchel, Geometriae Dedicata **64**: 277-282, 1997.

[Riv] Igor Rivin, Euclidean Structures on Simplicial Surfaces and Hyperbolic Volume, Ann. of Math, 2nd Ser., Vol. 139, No. 3. (May, 1994), pp.553-580.

[Spr] Springborn, B., A variational principle for weighted Delaunay triangulations and hyperideal polyhedra, arXiv: math.GT/0603097.