Print Form

DEPARTMENT OF MATHEMATICS

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year: Fall 2008

Name: McKenzie Lamb

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor
MATH 920	Dissertation	S	Philip Foth
LING 538	Computational Linguistics	A	Sandiway Foth

List the title, date and location of any talks you have given, either here or elsewhere:

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

See attached.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- [x] Independent Study
- [ ] Oral Comprehensive Exam
- [ ] Commence Thesis Research
- [ ] Conference attendance
- [ ] Conference participation
- [ ] Complete PhD

Professional development and outreach:

- [ ] AP Calculus Visit
- [ ] High School Workshops
- [ ] Undergraduate Research Project
- [ ] Undergraduate Research Seminar
- [x] Super TA
- [ ] Mentoring junior graduate students for the qualifying exams
- [ ] RTG (help organize)
- [ ] Research Seminar (help organize)

- [x] Other (please specify)

I have applied for a variety of academic jobs in mathematics.

Attach a brief statment about your academic progress and professional development during the period of support.

VIGRE REPORT: FALL 2008

MCKENZIE LAMB

Research

Primary Result. Using the Gram-Schmidt algorithm, it is possible to decompose the Lie group $G = SL_n(\mathbb{C})$ into the product $G = KAN$, where $K = SU(n)$, A is the set of diagonal matrices in G with real, positive diagonal, and N is the set of upper triangular matrices with each diagonal entry equal to one. More generally, for any complex, semisimple Lie group G, there exist decompositions–called Iwasawa decompositions–of the form $G = KAN$, where K is compact, A is abelian, and N is unipotent. For a given Iwasawa decomposition, the groups K and AN can be given multiplicative Poisson structures in such a way that they are mutually dual Poisson Lie groups ([LW90]). These Poisson structures are called the *standard* or Lu-Weinstein structures on each group. On the other hand, denoting the Lie algebras of G, K, and AN by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{a}+\mathfrak{n}$, respectively, the decomposition $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a}+\mathfrak{n})$ yields an identification of $\mathfrak{a}+\mathfrak{n}$ with $\mathfrak{k}^*$ via the imaginary part of the Killing form. Using this identification, the Lie bracket on $\mathfrak{k}$ induces a Poisson structure on $\mathfrak{k}^* \cong \mathfrak{a}+\mathfrak{n}$, called the *linear* Poisson structure. Ginzburg and Weinstein proved in [GW92] the existence of a Poisson isomorphism from $\mathfrak{g}^*$, endowed with the linear Poisson structure, to G^* endowed with its standard Poisson structure. For the case $G = SU(n)$, Flaschka and Ratiu conjectured the existence of a distinguished Ginzburg-Weinstein isomorphism which intertwines the Gelfand-Tsetlin coordinates on $\mathfrak{su}(n)^*$ and $SU(n)^*$ ([FR96]). This conjecture was later proved by Alekseev and Meinrenken in [AM07].

I am currently working with my advisor, Philip Foth, toward proving a similar result in which the compact group $SU(n)$ is replaced by the non-compact group $SU(p,q)$. In the $SU(p,q)$ case, we have only a partial decomposition at the group level. At the Lie algebra level, however, we have a full decomposition, as in the compact case, and this decomposition can be used to produce mutually dual multiplicative Poisson structures on $SU(p,q)$ and AN. This decomposition also yields a linear Poisson structure on $\mathfrak{su}(p,q)^* \cong \mathfrak{a}+\mathfrak{n}$. We hope to prove the existence of a Poisson isomorphism from $\mathfrak{su}(p,q)^*$ to $SU(p,q)^* \cong AN$, endowed with the linear and standard Poisson structures, respectively, induced by $SU(p,q)$. Ideally, this isomorphism will intertwine the Gelfand-Tsetlin coordinates (where they are defined).

For the case $p = q = 1$, we have succeeded in writing the required isomorphism in coordinates. For the higher dimensional cases, the proofs in [GW92] and [AM07] break down in the noncompact setting. In particular, the proof given in [GW92] employs a

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Moser-type argument which uses the fact that the symplectic leaves of the dual of $SU(n)$ (and more generally, the dual of a compact Poisson Lie group K) are compact. In the $SU(p,q)$ case, the symplectic leaves of AN are not compact, and the Ginzburg-Weinstein proof cannot be used. However, Evens and Lu have shown in [EL01] (building on a result of Drinfeld's) that each symplectic leaf of AN can be mapped in a one-to-one fashion onto an $SU(p,q)$ orbit in a certain compact space. We attempted to prove that the algebraic (Zariski) closures of these orbits are smooth. Our approach was to realize these closures as fixed point sets of involutions of smooth manifolds. Unfortunately, we were not able to find such involutions (and we are now uncertain whether they exist).

We have since turned to an alternative approach: Let $G_0 = SU(p,q)$. Each symplectic leaf can be identified with G_0/H for some subgroup $H \subset G_0$. The (partial) flag manifold G_0/H can be embedded as an open subset in K/H, where K is the compact group $SU(p+q)$. The algebraic closure of G_0/H in K/H is then all of K/H, a smooth, compact manifold. It only remains to show that there is a Poisson structure on K/H such that this embedding is Poisson. Put another way, it remains to be shown that the Poisson structure on G_0/H can be extended to all of K/H.

The compact space into which the symplectic leaves of AN can be mapped is the variety $\mathcal{L}$ of subalgebras of $\mathfrak{sl}(n,\mathbb{C})$ which are Lagrangian (half-dimensional, pair to 0 with themselves) with respect to the imaginary part of the Killing form (this is a symmetric form on $\mathfrak{sl}(n,\mathbb{C})$ of signature (n,n)). The space of Lagrangian *subspaces* of $\mathfrak{sl}(n,\mathbb{C})$ can be identified with $O(n^2-1)$, but the subalgebra requirement is much more difficult to handle. In the $n=2$ case, $\mathcal{L}$ has two connected components. The component containing $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, denoted by $\mathcal{L}_0$, can be identified with $SO(3) \cong \mathbb{R}P^3$. The other component can be identified with $\mathbb{C}P^1$. In the $SU(1,1)$ case, the image of each leaf in $\mathcal{L}_0$ can be realized as an open half of an ellipse. The topological closure is therefore diffeomorphic to a disk, and the algebraic closure is the entire ellipse. Both closures are obviously smooth. This sort of concrete realization does not seem to be possible in the higher-dimensional cases.

Auxiliary Results.

- I wrote a Maple program which allowed me to compute the standard Poisson structure on the group AN induced by $SU(3)$. I then wrote another program which pushed this structure forward under an isomorphism to H_3, the space of 3×3 Hermitian matrices.
- Let $G_0 = SU(p,q)$. The dressing orbits in AN are parameterized by points $\lambda \in \mathfrak{a}$. The stabilizer subgroup in G_0 of any *regular* λ is the torus $T \subset G_0$. Let π_λ denote the restriction of π_{AN} to the orbit Ψ_λ through the point $e^{-\lambda} \in A$. Then for a regular λ, identifying Ψ_λ with G_0/T, π_λ can be thought of as a Poisson structure on G_0/T. I computed an expression for this Poisson structure.

Professional Development

Vertical Integration. I served as the super-TA for the Math 323 course taught by Shankar Venkataramani. There was a great deal of frustration among students taking the course (as I am sure is typical), and I spent many hours working with them both in my office and during weekly problem sessions.

I also began developing labs for the math 215 course. This was to be a part of the vertical integration component of my VIGRE grant for the spring semester, but I wanted to have at least some labs completed early so that they could be used this semester. Jeff Taft (who is currently teaching the course) and I have developed two labs, both of which illustrate various geometric properties of linear transformations. One lab shows the image of a parallelogram in $\mathbb{R}^2$ under the action of a 2×2 matrix. Students can vary both the input vector and the matrix entries. This lab was created using the open-source software available at `www.geogebra.org`. The other lab illustrates similar concepts for linear transformations of $\mathbb{R}^3$. This lab makes use of the 3D graphing applet available at `http://www.pierce.ctc.edu/dlippman/g/`GrapherLaunch.html1. The primary purpose of both labs is to help students understand the relationships between the algebraic properties of matrices and the geometric properties of the associated linear maps.

Ph.D. Requirements. I completed the language examination in German with Dr. Klaus Lux.

Job Applications. I wrote my teaching and research statements, redesigned my web page, discussed my research with Dr. Flaschka (so that he could write a letter of recommendation), and completed many job applications.

References

[AM07] A. Alekseev and E. Meinrenken. Ginzburg-Weinstein via Gelfand-Zeitlin. *Journal of Differential Geometry*, 76(1):1–34, 2007.

[EL01] S. Evens and J.H. Lu. On the variety of Lagrangian subalgebras, I. *Annales scientifiques de l'Ecole normale supérieure*, 34(5):631–668, 2001.

[FR96] H. Flaschka and T. Ratiu. A convexity theorem for Poisson actions of compact Lie groups. *Annales Scientifiques de l'École Normale Supérieure Sér. 4*, 29(6):787–809, 1996.

[GW92] V.L. Ginzburg and A. Weinstein. Lie-Poisson Structure on Some Poisson Lie Groups. *Journal of the American Mathematical Society*, 5(2):445–453, 1992.

[LW90] J.H. Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Diff. Geom*, 31(2):501–526, 1990.