

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year:

Summer 2008

Name: Tom LaGatta

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor
	Discrete Spatial Processes in Probability		Grimmett
	Brownian Motion and Analysis		Burdzy

List the title, date and location of any talks you have given, either here or elsewhere:

Geodesics in a Random Riemannian Setting - UBC/PIMS Summer School in Probability - June 2008

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

Let $f(x)$ be a smooth, finite-correlation Gaussian random field on the plane \mathbb{R}^2 . Take the conformal Riemannian metric $e^{2f(x)} dx^2$, where dx^2 is the usual Euclidean inner product. By definition, geodesics are locally length-minimizing, but it's generally a difficult problem in geometry to study when they are globally length-minimizing. Fix a unit direction v , and consider the geodesic $x(t)$ from the origin in direction v . The problem I am working on is to show that $P[x(t) \text{ is globally length-minimizing}] = 0$. I show this by demonstrating a non-zero probability that $x(t)$ wanders into a patch of large positive curvature and stays there for long enough. To show this, I need a tightness condition: that as we translate along the geodesic $x(t)$, the distribution of the metric does not become too unreasonable. Most of my research progress this summer was working on this.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- Independent Study
- Oral Comprehensive Exam
- Commence Thesis Research
- Conference attendance
- Conference participation
- Complete PhD

Professional development and outreach:

- AP Calculus Visit
- High School Workshops
- Undergraduate Research Project
- Undergraduate Research Seminar
- Super TA
- Mentoring junior graduate students for the qualifying exams
- RTG (help organize)
- Research Seminar (help organize)

Other (please specify)

Attach a brief statement about your academic progress and professional development during the period of support.

VIGRE FINAL REPORT

TOM LAGATTA

1. SUMMER SCHOOL IN PROBABILITY

In June I attended the UBC/PIMS Summer School in Probability in Vancouver, Canada. I attended the two courses offered: Geoffrey Grimmett's "Discrete Spatial Processes in Probability" and Chris Burdzy's "Brownian Motion and Analysis." Both were very interesting but Grimmett's course had the most impact on my research. My research problem deals with continuum probability, but under suitable discretizations one can take advantage of techniques from discrete probability. At the summer school, I had many good discussions with Grimmett, who had worked on a number of discrete problems with certain similar characteristics and had a lot of insight on my particular problem.

One of the most notable techniques is finite energy, which for percolation, for example, says that if an event has positive probability, changing the value at one bond does not make the probability of the event zero. The situation is more complicated in the continuum—one cannot just change the value of a random field at one point—but the Gaussian structure of the field lets one still do something along these lines. The ideas have many names: mutual absolute continuity, Girsanov formula, Cameron-Martin transformation—but the basic idea is one of finite energy, that one is free to change the field locally without too much trouble.

2. RESEARCH PROGRESS

Let $f(x)$ be a smooth, finite-correlation Gaussian random field on the plane \mathbb{R}^2 . Take the conformal Riemannian metric $e^{2f(x)}dx^2$, where dx^2 is the usual Euclidean inner product. By definition, geodesics are locally length-minimizing, but it's generally a difficult problem in geometry to study when they are globally length-minimizing. Fix a unit direction v , and consider the geodesic $x(t)$ from the origin in direction v . The problem I am working on is to show that

$$P[x(t) \text{ is globally length-minimizing}] = 0.$$

I show this by demonstrating a non-zero probability that $x(t)$ wanders into a patch of large positive curvature and stays there for long enough. To show this, I need a tightness condition: that as we translate along the geodesic $x(t)$, the distribution of the metric does not become too unreasonable. In particular, if E_n is the event that $x(t)$ is length-minimizing up to the sphere $S(n)$ of radius n , and x_n is the hitting point of $S(n)$, I need to show tightness: for any ϵ , there exists a compact $K \in \mathcal{B}(\mathbb{R})$ such that

$$P[f(x_n) \in K | E_n] \geq 1 - \epsilon.$$

Date: 12 September 08.

If this is the case, then there is a subsequence n_k such that the distributions converge to a limiting distribution, and I can apply finite-energy methods on that limiting distribution to show that eventually the geodesic must cross into a positive-curvature region and lose its length-minimization property.

This summer I also did a lot of reading about random walk in random environment (RWRE). My problem is one of deterministic motion in random environment, but a lot of the ideas of RWRE can be translated to it. For example, there is a successful technique called “point of view of the particle” in RWRE. Take the lattice \mathbb{Z}^2 , and consider a (reasonable) family of random variables $\{\pi_{ij}\}_{i,j \in \mathbb{Z}^2}$ on the probability space (Ω, \mathcal{F}, P) . Then for each environment $\omega \in \Omega$, let X_n be a random walk with the transition probabilities $\{\pi_{ij}(\omega)\}$. The idea of point of view of the particle is to fix an initial environment ω , then at each time step translate it to a new environment $T^{X_n}(\omega)$, always assuming the particle stays at the origin. Thus instead of considering these two levels of probability, we treat this as a random walk $T^{X_n}(\omega)$ on the very large space Ω . Using these ideas, a number of authors including Varadhan, Sznitman, Zeitouni, Kozlov, and Rassoul-Agha have, under certain hypotheses, been able to prove laws of large numbers and central limit theorems for RWRE. While I was not able to directly apply their proofs to show tightness in my problem (mainly because their arguments focus on proving laws of large numbers, which I already have from my shape theorem result), I have a better understanding on the limits of their methods and how they might still be applicable in different settings.