Print Form

DEPARTMENT OF MATHEMATICS

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year: Summer 2008

Name: McKenzie Lamb

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor

List the title, date and location of any talks you have given, either here or elsewhere:

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

See attached.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- [ ] Independent Study
- [ ] Oral Comprehensive Exam
- [ ] Commence Thesis Research
- [x] Conference attendance
- [ ] Conference participation
- [ ] Complete PhD

Professional development and outreach:

- [ ] AP Calculus Visit
- [ ] High School Workshops
- [ ] Undergraduate Research Project
- [ ] Undergraduate Research Seminar
- [ ] Super TA
- [ ] Mentoring junior graduate students for the qualifying exams
- [ ] RTG (help organize)
- [ ] Research Seminar (help organize)

- [x] Other (please specify)

I participated in the integration workshop. (See attached.)

Attach a brief statment about your academic progress and professional development during the period of support.

VIGRE REPORT: SUMMER 2008

MCKENZIE LAMB

Research

Using the Gram-Schmidt algorithm, it is possible to decompose the Lie group $G = SL_N(\mathbb{C})$ into the product $G = KAN$, where $K = SU(n)$, A is the set of diagonal matrices in G with real, positive diagonal, and N is the set of upper triangular matrices with each diagonal entry equal to one. More generally, for any complex, semisimple Lie group G, there exist decompositions–called Iwasawa decompositions–of the form $G = KAN$, where K is compact, A is abelian, and N is unipotent. For a given Iwasawa decomposition, the groups K and AN can be given multiplicative Poisson structures in such a way that they are mutually dual Poisson Lie groups ([LW90]). These Poisson structures are called the *standard* or Lu-Weinstein structures on each group. On the other hand, denoting the Lie algebras of G, K, and AN by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{a} + \mathfrak{n}$, respectively, the decomposition $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$ yields an identification of $\mathfrak{a} + \mathfrak{n}$ with $\mathfrak{k}^*$ via the imaginary part of the Killing form. Using this identification, the Lie bracket on $\mathfrak{k}$ induces a Poisson structure on $\mathfrak{k}^* \cong \mathfrak{a} + \mathfrak{n}$, called the *linear* Poisson structure. Ginzburg and Weinstein proved in [GW92] the existence of a Poisson isomorphism from $\mathfrak{g}^*$, endowed with the linear Poisson structure, to G^* endowed with its standard Poisson structure. For the case $G = SU(n)$, Flaschka and Ratiu conjectured the existence of a distinguished Ginzburg-Weinstein isomorphism which intertwines the Gelfand-Tsetlin coordinates on $\mathfrak{su}(n)^*$ and $SU(n)^*$ ([FR96]). This conjecture was later proved by Alekseev and Meinrenken in [AM07].

I am currently working with my advisor, Philip Foth, toward proving a similar result in which the compact group $SU(n)$ is replaced by the non-compact group $SU(p, q)$. In the $SU(p, q)$ case, we have only a partial decomposition at the group level. At the Lie algebra level, however, we have a full decomposition, as in the compact case, and this decomposition can be used to produce mutually dual multiplicative Poisson structures on $SU(p, q)$ and AN. This decomposition also yields a linear Poisson structure on $\mathfrak{su}(p, q)^* \cong \mathfrak{a} + \mathfrak{n}$. We hope to prove the existence of a Poisson isomorphism from $\mathfrak{su}(p, q)^*$ to $SU(p, q)^* \cong AN$, endowed with the linear and standard Poisson structures, respectively, induced by $SU(p, q)$. Ideally, this isomorphism will intertwine the Gelfand-Tsetlin coordinates (where they are defined).

For the case $p = q = 1$, we have succeeded in writing the required isomorphism in coordinates. For the higher dimensional cases, the proofs in [GW92] and [AM07] break down in the noncompact setting. In particular, the proof given in [GW92] employs a Moser-type argument which uses the fact that the symplectic leaves of the dual of $SU(n)$

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(and more generally, the dual of a compact Poisson Lie group K) are compact. In the $SU(p,q)$ case, the symplectic leaves of AN are not compact, and the Ginzburg-Weinstein proof cannot be used. However, Evens and Lu have shown in [EL01] (building on a result of Drinfeld's) that each symplectic leaf of AN can be mapped in a one-to-one fashion onto an $SU(p,q)$ orbit in a certain compact space. The closure of this orbit is then a compact space which is (almost) a manifold with boundary. Moser-type arguments can by used on such spaces, and our goal is to do so in this case.

The compact space into which the symplectic leaves of AN can be mapped is the variety $\mathcal{L}$ of subalgebras of $\mathfrak{sl}(n,\mathbb{C})$ which are Lagrangian (half-dimensional, pair to 0 with themselves) with respect to the imaginary part of the Killing form (this is a symmetric form on $\mathfrak{sl}(n,\mathbb{C})$ of signature (n,n)). The space of Lagrangian *subspaces* of $\mathfrak{sl}(n,\mathbb{C})$ can be identified with $O(n^2-1)$, but the subalgebra requirement is much more difficult to handle. In the $n=2$ case, $\mathcal{L}$ has two connected components. The component containing $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$, denoted by $\mathcal{L}_0$, can be identified with $SO(3)\cong\mathbb{R}P^3$. The other component can be identified with $\mathbb{C}P^1$. A large portion of the summer was spent working out the geometry of $\mathcal{L}_0$, and of the $SU(1,1)$ orbits in $\mathcal{L}_0$.

Professional Development

Poisson Geometry Conference. From July 1st to 9th, I attended the "Poisson 2008" conference and school on Poisson geometry at the Ecole Polytechnique Fédérale de Lausanne (EPFL) in Switzerland. Information on the conference is available at http://www.math.unizh.ch/pois The school, in particular, was very helpful: Marius Crainic gave a mini-course covering basic Poisson geometry, and Marco Gualtieri gave an introduction to generalized complex structures. The other mini-courses were less accessible, but still interesting. At the conference proper, the talks by J.H. Lu and Henrique Bursztyn were the most useful to me.

Vertical Integration. I served as a senior graduate student at the integration workshop. Unfortunately, the workshop was postponed for one day due to a gas leak on August 8th, and I had to attend a funeral from the 10th through the 12th, so I was only able to attend on the 9th. I also designed a project on multilinear algebra for the workshop. The project will introduced the students to the basic notions in multilinear algebra, with a view toward preparing them to study differential forms in the geometry/topology core course.

Ph.D. Requirements. I have translated a portion of the (German) paper "Paramateresierungen von Konjugations Klassen in $\mathfrak{sl}(n)$," by Hanspeter Kraft. Dr. Klaus Lux is currently in the process of checking my translation. This will satisfy one of the communications requirements for the Ph.D. program.

References

[AM07] A. Alekseev and E. Meinrenken. Ginzburg-Weinstein via Gelfand-Zeitlin. *Journal of Differential Geometry*, 76(1):1–34, 2007.

[EL01] S. Evens and J.H. Lu. On the variety of Lagrangian subalgebras, I. *Annales scientifiques de l'Ecole normale supérieure*, 34(5):631–668, 2001.

[FR96] H. Flaschka and T. Ratiu. A convexity theorem for Poisson actions of compact Lie groups. *Annales Scientifiques de l'École Normale Supérieure Sér. 4*, 29(6):787–809, 1996.

[GW92] V.L. Ginzburg and A. Weinstein. Lie-Poisson Structure on Some Poisson Lie Groups. *Journal of the American Mathematical Society*, 5(2):445–453, 1992.

[LW90] J.H. Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Diff. Geom*, 31(2):501–526, 1990.