Print Form

DEPARTMENT OF MATHEMATICS

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year: Summer 2008

Name: Benjamin Pittman-Polletta

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor

List the title, date and location of any talks you have given, either here or elsewhere:

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

I completed an argument giving a geometric interpretation of and an explicit formula for the higher-order Hankel operators. These are conformally equivariant (differential) operators on spaces of holomorphic functions, parameterized by m. I am currently finalizing a paper on these results. It is available on my webpage.

I hope to use these operators to generate conformally invariant random functions on the disk. My first attempt at doing this necessitated factoring a determinant associated to a truncated Hankel operator of order m. In the case m=1, such a factorization exists as a consequence of the theory of the group of SU(2)-valued loops . After studying this case, I attempted to copy the structure of this factorization for m>1. Unfortunately, factorizations having this structure do not at present appear to exist.

In order to understand the case m=1 more deeply, and hopefully extend it to m>1, I am currently attempting to extend Prof. Pickrell's work on loops into SU(2). Namely, I am attempting to construct local factorizations for groups of loops into SU(n), or, more generally, groups of loops with values in a compact semisimple group K. This work involves calculations with the Weyl group, which I am in the process of automating.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- [x] Independent Study
- [ ] Oral Comprehensive Exam
- [x] Commence Thesis Research
- [x] Conference attendance
- [ ] Conference participation
- [ ] Complete PhD

Professional development and outreach:

- [ ] AP Calculus Visit
- [ ] High School Workshops
- [ ] Undergraduate Research Project
- [ ] Undergraduate Research Seminar
- [ ] Super TA
- [x] Mentoring junior graduate students for the qualifying exams
- [ ] RTG (help organize)
- [ ] Research Seminar (help organize)

- [ ] Other (please specify)

Attach a brief statment about your academic progress and professional development during the period of support.

VIGRE REPORT

BENJAMIN PITTMAN-POLLETTA

This past summer, I was the beneficiary of a VIGRE fellowship. In the three-month duration of the fellowship, I

The result I have gives explicit forms for a family of operator-valued maps on linear function spaces. I am interested in composing these maps with functionals to obtain measures on these function spaces. During May and June, I will be looking for conjectures about the existence and properties of these measures, in ways outlined below. During the late summer, I will undertake to prove these conjectures. At the same time, I will be studying the extension of my work to nonlinear spaces. Part of this has already been done by Prof. Pickrell [6]. I want to understand his results, and find new directions in which I can take them. I will begin pursuing these directions in the fall. A VIGRE fellowship, by providing me with a break from my teaching responsibilities, will be an invaluable aid to obtaining significant results in this short period of time.

1. Research

1.1. Finalizing Results on Higher-Order Hankel Operators.

At the beginning of the summer, my energies were concentrated on finalizing a result about higher-order Hankel operators. Higher-order Hankel operators are group-theoretic generalizations of classical Hankel operators; they arise as the irreducible subspaces of a particular space of operators under the action of $PSU(1,1)$. Last spring, I calculated what I conjectured were explicit forms for these operators (please see my VIGRE application for details). At the beginning of this summer, I attended a conference held in honor of Richard Rochberg, one of the mathematicians who was instrumental in developing the theory of higher-order Hankel operators. I talked with him and his collaborators in private twice during the conference. I conveyed my perspective on higher-order Hankel operators, and they helped me to see how my work connects to previous work in the area. Over this period, and after returning from the conference, Prof. Pickrell and I developed a conceptual interpretation of these operators which allowed us to prove they have the conjectured form.

The conceptual interpretation is the following. The space $H^0(\Delta)$ acts on $\Omega^{1/2}(S^1)$ via multiplication, and a classical Hankel operator is obtained by compressing this action. Namely, if $x \in H^0$, then

$$B_0(x) = \mathcal{P}_+ M_x \mathcal{P}_-$$

is the Hankel operator associated to x. Now, the Lie algebra of holomorphic vector fields on the circle is dense in $H^{-1}(\Delta)$, and elements of this dense

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subspace act on $\Omega^{1/2}(S^1)$. Specifically, they act as differential operators of order 1 via the Lie derivative. Compressing this action gives a Hankel operator of order 2.

This action in turn induces an action of $\mathcal{U}\left(H^{-1}\right)$, the universal enveloping algebra of the Lie algebra H^{-1}, on $\Omega^{1/2}(\Delta)$. At the beginning of the summer, I showed that $H^{-s}(\Delta)$ can be embedded in $\mathcal{U}\left(H^{-1}\right)$ in a $PSU(1,1)$-equivariant fashion. Thus, the action of the universal enveloping algebra induces an action of H^{-s} on $\Omega^{1/2}(S^1)$. In addition, the elements of H^{-s} act as differential operators of order $\leq s$. This proves that the higher-order Hankel operators have the conjectured form, namely they are compressions of this action.

The details of this result are now available in a paper on my webpage.

1.2. **Attempts to Factor Associated Determinants.** As discussed in my VIGRE application, I hope to use higher-order Hankel operators to construct conformally-invariant measures on the spaces $H^m(\Delta)$. These measures have heuristic densities given by the functional

$$\det\left(1 + B_s(x)B_s(x)^*\right),$$

where $B_s(x)$ is the Hankel operator of order $s+1$ with symbol x.

In the case $s = 0$, the existence of a measure with this density uses the fact that there is a change of variables which "diagonalizes" the above determinant. Namely, there are variables $\zeta_1, \zeta_2, \zeta_3, \ldots$, depending nonlinearly on the Fourier coefficients $x_1, x_2, x_3, \ldots$ of $x(z)$, such that

$$\det\left(1 + B_s(x)B_s(x)^*\right) = \prod_{i=1}^{\infty}(1 + |\zeta_i|^2)^i. \tag{1}$$

In July, I attempted to find analogous factorizations for the higher-order cases.

After studying the determinants in question in the cases $s = 0, 1, 2, 3$, certain patterns revealed themselves. First, for odd s, the above determinant appears to be a square. Second, the "straight" terms, i.e. terms which are powers of $|x_i|$, suggest the factorization (1), and also suggest the factorization

$$\det\left(1 + B_1(x)B_1(x)^*\right) = \left(\prod_{i=1}^{\infty} \sum_{\substack{n+i \text{ odd} \\ 0 \leq n \leq \frac{i+1}{2}}} (1 + n^2|\zeta_i|^2)\right),$$

and other similar factorizations for $s > 1$. This, along with the triangular structure of the diagonalization for $s = 0$, gave me something concrete to shoot for. Unfortunately, factorizations with this structure do not appear to exist. In the case $s = 1$, they exist for small truncations of the Hankel operators, but even then they are not unique.

1.3. **Extending Results on the Affine Lie Group $\hat{SU}(2)$.** In August, in order to better understand the case $s = 0$ and perhaps develop another way to take advantage of the regularities apparent in the determinants

$\det(1 + B_s(x)B_s(x)^*)$, I began working with Doug Pickrell to extend results on the group of $SU(2)$-valued loops, the affine group $\hat{SU}(2)$. There is a local factorization of this group which implies the diagonalizing change of variables in the case $s = 0$. This local factorization is an extension to infinite dimensions of work of Lu and Evens. The factorization for a finite-dimensional compact semisimple group K depends on a choice of minimal factorization of the longest Weyl group element,

$$w_0 = r_1 r_2 r_3 \dots r_m,$$

where the r_i are simple reflections and m is the number of positive roots in $\mathfrak{h}$, the Cartan subalgebra of $\mathfrak{k}$. In the affine case, there is no longest Weyl group element, but an element $r_1 r_2 \dots$ of the Weyl group may still be minimal if $r_1 \dots r_k$ is minimal for all k. The key in extending the factorization to the affine case is finding a minimal element that flips all the positive roots of the affine algebra. We have found such elements in the cases $SU(3)$, G_2, and $SU(4)$, but we have yet to find a generalizable pattern. Currently, I am seeking to automate the lengthy Weyl group calculations this research involves, and to obtain a more geometric perspective on the actions of the Weyl group elements in question.

2. Background Reading

This summer was also a good opportunity to strengthen my background in both representation theory and mathematical physics. To review, I read the book ([3]) over the summer. I also worked through ([5]), a very good introduction to the Virasoro algebra, affine Lie algebras, and their applications in physics. With more difficulty, I read through the first few chapters of ([4]), a more thorough treatment of Kac-Moody algebras. I also read a number of seminal and more recent papers in conformal field theory and conformal probability (e.g. [1], [2]), which have helped me to understand the connections between conformal field theory, the stochastic Loewner evolution, and the representations of the Virasoro algebra. I have been discussing these areas with Tom Kennedy, and I will talk about them in the Geometry Seminar on September 23.

3. Service & Professional Development

As stated above, I traveled to a conference at the beginning of the summer, where I made contacts in the fields of operator theory, complex analysis, and lie theory. I continue to correspond with these contacts regarding my results on higher-order Hankel operators. In Tucson, during June and July, I mentored graduate students in the Interdisciplinary Program for Applied Mathematics preparing to take their qualifying exams in August. We held study sessions every Tuesday morning, where we went over the solutions to past qualifying exams and reviewed the material on the test. This was enjoyable for me, and I believe it was useful for them.

References

[1] Belavin A.A., Polyakov A.M., Zamolodchikov A.B. "Infinite conformal symmetry of critical fluctuations in two dimensions." J. Statistical Physics, vol. 34, nos. 5/6 (1984) pp.763-774.

[2] Friedrich, R. & Werner W. "Conformal fields, restriction properties, degenerate representations and SLE." C. R. Mathematiques, vol. 335, no. 11 (2002) pp.947-952.

[3] Hall, B.C. *Lie Groups, Lie Algebra, and Representations.* Springer-Verlag, NY. 2003.

[4] Kac, V. *Infinite-Dimensional Lie Algebras* Cambridge University Press, Cambridge. 1990.

[5] Kac, V. & Raina, A.G. *Bombay Lecturs on Highest Weight Representations of Infinite-Dimensional Lie Algebras.* Advanced Series in Mathematical Physics, Vol. 2. World Scientific, New York. 1998.

[6] Pickrell D. "A survey of conformally invariant measures on $H^m(\triangle)$." arXiv:math/0702672.

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