

DEPARTMENT OF MATHEMATICS

# VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year:

Spring 2009

Name: McKenzie Lamb

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor
Math 920	Dissertation	S	Philip Foth

List the title, date and location of any talks you have given, either here or elsewhere:

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

See attached.

List publications, if any.

Check all activities you completed during the funded period:

Academics:

- Independent Study
- Oral Comprehensive Exam
- Commence Thesis Research
- Conference attendance
- Conference participation
- Complete PhD

Professional development and outreach:

- AP Calculus Visit
- High School Workshops
- Undergraduate Research Project
- Undergraduate Research Seminar
- Super TA
- Mentoring junior graduate students for the qualifying exams
- RTG (help organize)
- Research Seminar (help organize)

Other (please specify)

Attach a brief statment about your academic progress and professional development during the period of support.

# VIGRE REPORT: SPRING 2009

MCKENZIE LAMB

## RESEARCH

Using the Gram-Schmidt algorithm, it is possible to decompose the Lie group  $G = SL_N(\mathbb{C})$  into the product  $G = KAN$ , where  $K = SU(n)$ ,  $A$  is the set of diagonal matrices in  $G$  with real, positive diagonal, and  $N$  is the set of upper triangular matrices with each diagonal entry equal to one. More generally, for any complex, semisimple Lie group  $G$ , there exist decompositions—called Iwasawa decompositions—of the form  $G = KAN$ , where  $K$  is compact,  $A$  is abelian, and  $N$  is unipotent. For a given Iwasawa decomposition, the groups  $K$  and  $AN$  can be given multiplicative Poisson structures in such a way that they are mutually dual Poisson Lie groups ([LW90]). These Poisson structures are called the *standard* or Lu-Weinstein structures on each group. On the other hand, denoting the Lie algebras of  $G$ ,  $K$ , and  $AN$  by  $\mathfrak{g}$ ,  $\mathfrak{k}$ , and  $\mathfrak{a} + \mathfrak{n}$ , respectively, the decomposition  $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$  yields an identification of  $\mathfrak{a} + \mathfrak{n}$  with  $\mathfrak{k}^*$  via the imaginary part of the Killing form. Using this identification, the Lie bracket on  $\mathfrak{k}$  induces a Poisson structure on  $\mathfrak{k}^* \cong \mathfrak{a} + \mathfrak{n}$ , called the *linear* Poisson structure. Ginzburg and Weinstein proved in [GW92] the existence of a Poisson isomorphism from  $\mathfrak{g}^*$ , endowed with the linear Poisson structure, to  $G^*$  endowed with its standard Poisson structure. For the case  $G = SU(n)$ , Flaschka and Ratiu conjectured the existence of a distinguished Ginzburg-Weinstein isomorphism which intertwines the Gelfand-Tsetlin coordinates on  $\mathfrak{su}(n)^*$  and  $SU(n)^*$  ([FR96]). This conjecture was later proved by Alekseev and Meinrenken in [AM07].

I am currently working with my advisor, Philip Foth, toward proving a similar result in which the compact group  $SU(n)$  is replaced by the non-compact group  $SU(p, q)$ . In the  $SU(p, q)$  case, we have only a partial decomposition at the group level. At the Lie algebra level, however, we have a full decomposition, as in the compact case, and this decomposition can be used to produce mutually dual multiplicative Poisson structures on  $SU(p, q)$  and  $AN$ . This decomposition also yields a linear Poisson structure on  $\mathfrak{su}(p, q)^* \cong \mathfrak{a} + \mathfrak{n}$ . We hope to prove the existence of a Poisson isomorphism from  $\mathfrak{su}(p, q)^*$  to  $SU(p, q)^* \cong AN$ , endowed with the linear and standard Poisson structures, respectively, induced by  $SU(p, q)$ . Ideally, this isomorphism will intertwine the Gelfand-Tsetlin coordinates (where they are defined).

For the case  $p = q = 1$ , we have succeeded in writing the required isomorphism in coordinates. For the higher dimensional cases, the proofs in [GW92] and [AM07] break down in the noncompact setting. In particular, the proof given in [GW92] employs a Moser-type argument which uses the fact that the symplectic leaves of the dual of  $SU(n)$

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(and more generally, the dual of a compact Poisson Lie group  $K$ ) are compact. In the  $SU(p, q)$  case, the symplectic leaves of  $AN$  are not compact, and the Ginzburg-Weinstein proof cannot be used. However, Evens and Lu have shown in [EL01] (building on a result of Drinfeld's) that each symplectic leaf of  $AN$  can be mapped in a one-to-one fashion onto an  $SU(p, q)$  orbit in a certain compact space. We attempted to prove that the algebraic (Zariski) closures of these orbits are smooth. Our approach was to realize these closures as fixed point sets of involutions of smooth manifolds. Unfortunately, we were not able to find such involutions (and we are now uncertain whether they exist).

We have since turned to an alternative approach: Let  $G_0 = SU(p, q)$ . Each coadjoint (dressing) orbit can be identified with  $G_0/H$  for some subgroup  $H \subset G_0$ . The (partial) flag manifold  $G_0/H$  can be embedded as an open subset in  $K/H$ , where  $K$  is the compact group  $SU(p+q)$ . The algebraic closure of  $G_0/H$  in  $K/H$  is then  $K/H$ , a smooth, compact manifold. We have shown that for the regular orbits (in which case  $H$  is the standard torus) in the  $SU(1, 1)$  and  $SU(1, 2)$  cases, there is a Poisson structure on  $K/H$  such that this embedding is Poisson.

Over the course of the spring semester, I have typed these results as well as the majority of the background material for my dissertation. My final defense will take place in August.

#### PROFESSIONAL DEVELOPMENT

**Vertical Integration.** I served as the super-TA for the Math 323 course taught by Fred Stevenson.

**Ph.D. Requirements.** I spoke on January 28th at Sahuaro High School and on April 14th in the Geometry Seminar. With these talks, I completed the communications requirement for the Ph.D. program.

**Job Applications.** In January, I attended the Joint Meetings of the AMS in Washington D.C., where I had interviews with several schools. My interview with Saint Francis University in Pennsylvania led to a phone interview, followed by an on-campus interview, and finally a job offer. I also had an on-campus interview at Ripon College in Wisconsin. This interview also led to a job offer, which I ultimately accepted. I will begin a tenure-track position at that school on August 7th.

#### REFERENCES

- [AM07] A. Alekseev and E. Meinrenken. Ginzburg-Weinstein via Gelfand-Zeitlin. *Journal of Differential Geometry*, 76(1):1–34, 2007.
- [EL01] S. Evens and J.H. Lu. On the variety of Lagrangian subalgebras, I. *Annales scientifiques de l'École normale supérieure*, 34(5):631–668, 2001.
- [FR96] H. Flaschka and T. Ratiu. A convexity theorem for Poisson actions of compact Lie groups. *Annales Scientifiques de l'École Normale Supérieure Sér. 4*, 29(6):787–809, 1996.

- [GW92] V.L. Ginzburg and A. Weinstein. Lie-Poisson Structure on Some Poisson Lie Groups. *Journal of the American Mathematical Society*, 5(2):445–453, 1992.
- [LW90] J.H. Lu and A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. *J. Diff. Geom.*, 31(2):501–526, 1990.