Print Form

DEPARTMENT OF MATHEMATICS

VIGRE Funding Report

(due 30 days after semester of support)

Semester/Summer and Year: Spring 2009

Name: Benjamin Pittman-Polletta

List the graduate courses you have taken this semester (including independent studies), your grades, and the instructors:

Course	Title	Grade	Instructor

List the title, date and location of any talks you have given, either here or elsewhere:

If you are working on your dissertation, include a one paragraph description of your research progress. If you have not yet begun dissertation research, describe your progress toward finding a dissertation topic and advisor and beginning that research.

I am studying factorization in unitary loop groups. A loop group is an infinite-dimensional group whose elements are smooth loops in a finite-dimensional Lie group. Multiplication is pointwise. I am studying a new kind of factorization on loops into unitary groups, which I call the root-space factorization, already well understood for the group of loops into SU(2). During the spring semester, I studied the root-space factorization on loops into SU(3), and generalized many of my results to the group of loops into SU(n). These included a characterization of certain types of factorizable loops, as well as an algorithmic approach to factoring loops. These results allowed Professor Doug Pickrell and I to write a paper about root-space factorization in an arbitrary unitary group, now posted to the arXiv.

List publications, if any.

"Unitary loop groups and factorization." Pickrell D. & Polletta B. arXiv:0905.2911v1 (2009) "A geometric interpretation and explicit form for higher-order Hankel operators." Pittman-Polletta B. arXiv:0901.2953 (2009)

Check all activities you completed during the funded period:

Academics:

- ☐ Independent Study
- ☐ Oral Comprehensive Exam
- ☐ Commence Thesis Research
- ☐ Conference attendance
- ☐ Conference participation
- ☐ Complete PhD

Professional development and outreach:

- ☐ AP Calculus Visit
- ☐ High School Workshops
- ☐ Undergraduate Research Project
- ☐ Undergraduate Research Seminar
- ☐ Super TA
- ☐ Mentoring junior graduate students for the qualifying exams
- ☐ RTG (help organize)
- ☐ Research Seminar (help organize)

☐ Other (please specify)

Attach a brief statment about your academic progress and professional development during the period of support.

VIGRE REPORT

BENJAMIN PITTMAN-POLLETTA

I received a VIGRE fellowship during the spring semester of the 2008-2009 academic year, my fifth year as a graduate student at the University of Arizona. I spent that time working on dissertation research under the supervision of Professor Doug Pickrell. The following is a report on my activities; please see my VIGRE proposal for details.

1. Research

1.1. **Lifted SLE.** At the beginning of the semester, I spent several weeks investigating an idea proposed by Bauer and Bernard ([1],[2],[3]), heuristically relating the stochastic Loewner evolution (SLE) to representations of the Virasoro algebra (Vir) which are degenerate at level two. Bauer and Bernard propose constructing a stochastic differential equation on a space of conformal maps, which they call a "Virasoro group". This Virasoro group is the formal group obtained by exponentiating the negative root spaces of Vir, and Vir acts on it as a space of vector fields. The stochastic differential equation (SDE) these authors propose on the Virasoro group, called "lifted SLE", is heuristically generated by a null vector at level two in the representation of Vir on the Virasoro group. Lifted SLE is closely related to SLE.

I am interested in making lifted SLE rigorous, in generating other SDEs using null vectors in the representation of Vir at other levels, and in exploring the growth processes related to the resulting flows of conformal maps. After exploring the literature, I found other papers by physicists dealing with SDEs generated by other null vectors ([5],[?]), but they run into a fundamental problem. The connection between level two null vectors and SLE is related to the fact that SLE is driven by a scaled Brownian motion $\sqrt{\kappa}B(t)$. Brownian motion satisfies the equation

$$(dB)^2 = dt,$$

and this is precisely the property which allows an SDE driven by Brownian motion to be related to a null vector at level two. Relating an SDE to a null vector at level n would require a process X satisfying

$$(dX)^n = dt.$$

1.2. **Factorization in Unitary Loop Groups.** The rest of the semester I worked with Prof. Pickrell to generalize results on factorization in unitary loop groups. Prof. Pickrell obtained results for the group of loops into $SU(2)$, denoted $LSU(2)$ ([6],[7]). This semester, we extended these results to the group of loops into an arbitrary compact, simple Lie group K, denoted LK [8].

In the case of $LSU(2)$, for $n \in \mathbb{N}$, define the maps k_n^+ and k_n^- from $\mathbb{C}$ into $LSU(2)$,

$$k_n^- : \eta \longmapsto \frac{1}{\sqrt{1+|\eta|^2}} \begin{pmatrix} 1 & \eta z^n \\ -\bar{\eta} z^{-n} & 1 \end{pmatrix}, \qquad k_n^+ : \zeta \longmapsto \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} 1 & \zeta z^{-n} \\ -\bar{\zeta} z^n & 1 \end{pmatrix}.$$

Next, define the maps k_- and k_+ from the Schwartz space of rapidly decreasing sequences $\mathcal{S}(\mathbb{Z}, \mathbb{C})$ into $LSU(2)$,

$$k_- : \{\eta_n\}_0^\infty \longmapsto \dots k_n^-(\eta_n) \dots k_0^-(\eta_0), \qquad k_+ : \{\zeta_n\}_1^\infty \longmapsto \dots k_n^+(\zeta_n) \dots k_1^+(\zeta_1).$$

Finally, define the map into $LSU(2)$,

$$\left\{\eta_j \in \mathbb{C}; \xi_n \in \mathbb{C}; \zeta_i \in \mathbb{C} \middle| j \in \mathbb{N}; n \in \mathbb{Z}; i \in \mathbb{N}\right\} \longmapsto k_-(\eta)^* \begin{pmatrix} \exp\left(\sum_n \xi_n z^n\right) & 0 \\ 0 & \exp\left(-\sum_n \xi_n z^n\right) \end{pmatrix} k_+(\zeta).$$

It turns out that every $k \in LSU(2)$ with a Riemann-Hilbert factorization is in the image of the above map. In other words, if k has a Riemann-Hilbert factorization, then it can be factored into an infinite product of the type above, for some coefficients η_j, ξ_n, ζ_i. I call such a factorization a root-space factorization.

Date: June, 2009.

During the first half of the semester, I worked out the details of the root-space factorization in $LSU(3)$ and $LSU(4)$. These examples allowed Prof. Pickrell and I to characterize the images of the maps k_- and k_+. For example, k is in the image of k_+ if and only if k has a triangular factorization and, for each fundamental representation π of K with lowest weight vector v, $\pi\left(k^{-1}\right)v$ is holomorphic, and its value at infinity is a real positive multiple of v.

I am currently attempting to deduce more transparent conditions ensuring that $k \in LSU(n)$ has a triangular factorization, given that k satisfies the above condition on its representations. My main tool is an iterative algorithm I designed last semester. This algorithm retrieves the coefficients ζ_n from the product $k_+(\zeta) \in LSU(n)$, and requires looking at all the fundamental representations of $SU(n)$ simultaneously.

For arbitrary K, the maps $k_n^{\pm}$ are determined by a particular type of periodic walk in the affine Weyl group corresponding to LK. Last semester, I realized that such a walk exists for all simple, compact K. Designing the iterative algorithm above led me to construct a specific such walk in the case of $SU(n)$. I am currently attempting to understand these walks on a deeper and more general level. For example, how many factorizations are there for a given periodic walk? There are determinant formulas for the number of walks remaining in a bounded region in an affine Weyl group [4], and these will certainly be of use.

Finally, in the case of $SU(n)$, I showed that the correspondence between the factorizations

$$k = lau, \qquad k = k_-^* \lambda k_+$$

can be interpreted as the result of performing a modified Gram-Schmidt algorithm on l, resulting in a factorization

$$l = k_-^* \lambda b.$$

Then, bau is in the image of k_+.

2. Professional Development

This Spring, I engaged in a number of professional development activities. I submitted my paper on Hankel operators [9] to the Rocky Mountain Journal of Mathematics. I attended several talks at both Entropy and the Quantum, a conference held at the University of Arizona, and the SAGE workshop on Ricci flow on homogeneous spaces, held at the University of Texas in Austin. I acted as a Super TA for Principles of Analysis. I also co-founded and co-organized the first semester of the Graduate College of Science's Science Salon.

The Science Salon is a informal, biweekly gathering of graduate students from all areas of science, designed to promote the open exchange of ideas on topics outside the traditional domain of the sciences. Seven Salons were held last semester. They touched on topics including randomness in scientific theory, progress in the social sciences, the role of arrogance in scientific culture, poetry and scientific education, and equality in the sciences. They were attended by groups of twenty to thirty graduate students, faculty members, and undergraduates. Participants were from a variety of departments, including Mathematics, Geography, Chemistry, Materials Science, Biology, and Physics. The Science Salon will continue in the fall.

3. Seminars, Colloquia, and Courses

As of Fall 2008, I completed the Applied Math program's course requirements. This semester, I did not take any courses for credit, but I sat in on Topics in Geometry, a course taught by David Glickenstein, dealing with the recent use of Ricci flow in the proof of the Poincare conjecture. I also attended the Geometry Seminar, the Mathematical Physics Seminar, and the Math and Applied Math Colloquia.

References

[1] Bauer M. & Bernard D. "SLE martingales and the Virasoro algebra." Physics Letters B 557 (2003) 309-316.
[2] Bauer M. & Bernard D. "SLE$_\kappa$ growth processes and conformal field theories." arXiv:math-ph/0206028v3
[3] Bauer M. & Bernard D. "Conformal field theories of stochastic Loewner evolutions." Commun. Math. Phys. 239 (2003) 493-521.
[4] Grabiner D.J. "Random walk in an alcove of an affine Weyl group, and non-colliding random walks on an interval." J Comb. Theory A 97 no. 2 (2002) 285-306.
[5] Lesage F. & Rasmussen J. "SLE-type growth processes and the Yang-Lee singularity." J. Math. Physics 45 no. 8 (2004) 3040-3048.
[6] Pickrell D. "Homogeneous Poisson structures on loop spaces of symmetric spaces." SIGMA 4 no. 69 (2008)
[7] PIckrell D. "Loops in SU(2) and factorization." arXiv:0903.4983v2 (2009)

[8] Pickrell D. & Polletta B. "Unitary loop groups and factorization." arXiv:0905.2911v1 (2009)
[9] Pittman-Polletta B. "A geometric interpretation and explicit form for higher-order Hankel operators." arXiv:0901.2953 (2009)

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