

**INTEGRATION WORKSHOP 2003**  
**TOPOLOGY EXERCISES**

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1. BASICS

**1.1. Continuity at a point.** Find a definition of “ $f : X \rightarrow Y$  is continuous at  $p \in X$ ” such that  $f$  is continuous if and only if it is continuous at each  $p \in X$ .

**1.2. Continuity and  $\epsilon$ - $\delta$ -continuity.** Let  $X$  and  $Y$  be metric spaces. A function  $f : X \rightarrow Y$  is said to be  $\epsilon$ - $\delta$ -continuous at  $p \in X$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $d_X(p, q) < \delta$  implies  $d_Y(f(p), f(q)) < \epsilon$ . One says that  $f$  is  $\epsilon$ - $\delta$ -continuous if it is  $\epsilon$ - $\delta$ -continuous at every  $p \in X$ . Prove that if  $X$  and  $Y$  are given the metric topologies then  $f$  is  $\epsilon$ - $\delta$ -continuous if and only if it is continuous in the topological sense.

**1.3. Bases and subbases.** A *basis* for a topology on  $X$  is a collection of subsets such that the open sets of  $X$  are precisely the unions of sets in the collection. A *subbasis* is a collection of subsets such that the finite intersections of sets in the collection form a basis.

Prove that for any collection of subsets of a set  $X$ , there is a unique topology on  $X$  for which the given collection is a subbasis.

The existence part of the analogous statement is false if we replace “subbasis” with “basis.” Find a set of axioms which characterize collections of subsets which are bases for a topology.

**1.4. Products.** Let  $X_\alpha (\alpha \in I)$  be a collection of topological spaces indexed by a set  $I$  (which is not assumed to be finite, or even countable). We define a topology on the Cartesian product  $X = \prod_{\alpha \in I} X_\alpha$  as follows: a basis is given by sets of the form  $\prod U_\alpha$  where  $U_\alpha \subset X_\alpha$  is open and for all but finitely many  $\alpha$ ,  $U_\alpha = X_\alpha$ . Note that if  $I$  is infinite this is not what you might guess at first. The rationale for this definition is that it is the smallest topology (fewest open sets) such that the projections  $X \rightarrow X_\alpha$  are continuous. Prove this.

Prove that the metric topology on  $\mathbb{R}^{m+n}$  is the product of the metric topologies on  $\mathbb{R}^m$  and  $\mathbb{R}^n$ .

Prove that  $\prod_{a \in [0,1]} [0, 1]$  (the product of copies of the unit interval indexed by the unit interval) with its product topology is not first countable.

**1.5. Infinitude of primes, topologically.** This exercise yields a topological proof (due to Furstenberg) that there are infinitely many prime numbers. By definition, an *arithmetic progression* is a set of integers of the form  $\{\dots, a - 2n, a - n, a, a + n, a + 2n, \dots\} = \{a + kn | k \in \mathbb{Z}\}$  for some  $a \in \mathbb{Z}$  and some non-zero  $n \in \mathbb{Z}$ . Define a topology on  $\mathbb{Z}$  by declaring that the open sets are the empty set and unions of arithmetic progressions. Check that this is indeed a topology. Prove that an arithmetic progression is also a closed set in this topology. Now prove that if there were

only finitely many primes, the set  $\{-1, 1\}$  would be open. Show that this set is not open and conclude that there are infinitely many primes.

Of course, there are other, easier proofs of this fact, but people were interested in this one because they hoped it would lead to a new proof of Dirichlet's theorem on primes in an arithmetic progression. That hope has yet to be born out.

**1.6. Topological groups.** A *topological group* is a Hausdorff topological space with a group structure (multiplication, inversion, identity) such that the multiplication and inversion maps  $G \times G \rightarrow G$  and  $G \rightarrow G$  are continuous. (Use the product topology on  $G \times G$ .) Homomorphisms of topological groups are required to be continuous and subgroups are given the induced topology.

Show that if  $H$  is a subgroup of  $G$ , then so is its closure  $\overline{H}$ . If  $H$  is normal then so is  $\overline{H}$ .

Show that an open subgroup is automatically also closed, but that a closed subgroup need not be open.

Prove that if  $H$  is a closed normal subgroup, then  $G/H$  (with the quotient topology) is a topological group.

**1.7. Exotic quotients and induced.** Quotient constructions can sometimes give rise to exotic topologies "in nature." For example, let  $X = \mathbb{R}^n$  and let  $G = GL_n(\mathbb{R})$ , the group of  $n \times n$  invertible matrices, act on  $X$  as usual, by linear transformations. Let  $Y$  be the space of orbits, so we have a natural map  $X \rightarrow Y$  sending a point to its orbit. Give  $X$  its usual (metric) topology and give  $Y$  the quotient topology. Describe completely the space  $Y$ .

Induced topologies can also be strange. If you already know about connectedness and compactness, try this (if not, come back to it after the next two lectures): Let  $X$  be the set of rational number between 0 and 1, i.e.,  $X = [0, 1] \cap \mathbb{Q}$ . Discuss the connectedness and compactness of this space.

**1.8. Locally closed sets.** A subset  $A$  of a topological space  $X$  is called *locally closed* if every  $p \in A$  has a neighborhood  $U_p$  such that  $A \cap U_p$  is closed in  $U_p$ . Prove that a set is locally closed if and only if it is the intersection of an open set and a closed set, if and only if it is the difference of two closed sets (i.e., of the form  $C_1 \setminus C_2$  where  $C_1$  and  $C_2$  are closed), if and only if it is open in its closure.

## 2. COMPACTNESS

**2.1. Criterion for homeomorphism.** Show that if  $X$  is compact and  $Y$  is Hausdorff and  $f : X \rightarrow Y$  is a continuous bijection, then  $f$  is a homeomorphism.

**2.2. Sequences.** Let  $X$  be a topological space and  $\{x_n | n = 1, 2, 3, \dots\}$  be a sequence in  $X$ . Define the notion of an accumulation point of the sequence. Prove that if  $X$  is compact, then every sequence has an accumulation point. Hint: First reduce to the case where infinitely many of the  $x_n$  are distinct. Argue that if the sequence has no accumulation point then it is discrete in the induced topology and closed as a subset of  $X$ , therefore compact. But a discrete space is compact if and only if it is finite.

**2.3. Compactness in metric spaces.** A metric space  $X$  is said to be *sequentially compact* if every sequence has a convergent subsequence. It is said to be *complete* if every Cauchy sequence converges to an element of  $X$  and it is said to be *totally bounded* if for every  $\epsilon > 0$ ,  $X$  can be covered with a finite number of balls of radius  $\epsilon$ . Prove that (i)  $X$  (with its metric topology) is compact if and only if (ii)  $X$  is sequentially compact if and only if (iii)  $X$  is complete and totally bounded. Hints: For (i) $\Rightarrow$ (ii) argue as in the previous exercise. For (ii) $\Rightarrow$ (iii), completeness is easy; for total boundedness, argue by contradiction: choose an  $\epsilon$  for which total boundedness fails and construct a sequence without an accumulation point. For (iii) $\Rightarrow$ (ii), cover  $X$  by finitely many 1-balls and argue that one of them contains infinitely many  $x_n$ ; cover this 1-ball by finitely many  $\frac{1}{2}$ -balls and argue  $\dots$ . For (ii) $\Rightarrow$ (i), first use (ii) $\Rightarrow$ (iii) to construct a dense sequence in  $X$  and use it to reduce to the case of a countable cover  $U_1, U_2, \dots$ . If this has no finite subcover, consider the sets  $C_n = X \setminus (U_1 \cup \dots \cup U_n)$  to construct a sequence and use a convergent subsequence to contradict the fact that the  $U_n$  cover  $X$ .

**2.4. Topological vector spaces.** A (real) *topological vector space* is a vector space  $V$  over  $\mathbb{R}$  with a Hausdorff topology such that the addition and scalar multiplication maps  $V \times V \rightarrow V$  and  $\mathbb{R} \times V \rightarrow V$  are continuous.

Prove that a linear map  $V \rightarrow W$  of topological vector spaces is continuous if and only if it is continuous at the origin.

If  $V$  is a finite dimensional vector space and we fix a vector space isomorphism  $\phi : \mathbb{R}^n \rightarrow V$ , then we may transfer the metric topology from  $\mathbb{R}^n$  to  $V$ : the open subsets of  $V$  are the sets  $\phi(U)$  where  $U$  is open in  $\mathbb{R}^n$ . Prove that the resulting topology on  $V$  is independent of the choice of  $\phi$ .

More generally, there is only one topology on a finite dimensional vector space  $V$  which makes it into a topological vector space. Hints for a proof: Choose a vector space isomorphism  $\phi : \mathbb{R}^n \rightarrow V$ . Define  $S = \{x \in \mathbb{R}^n \mid |x| = 1\}$  and  $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$ .

- (1) Prove directly from the definitions that  $\phi$  is continuous.
- (2) Prove that  $\phi(S)$  is closed in  $V$  and so  $V \setminus \phi(S)$  is open.
- (3) Use the continuity of  $\mathbb{R} \times V \rightarrow V$  to show that every neighborhood of the origin in  $V$  contains an open set  $U$  such that  $0 \in U$  and  $\alpha U \subset U$  for all real  $-1 < \alpha < 1$ .
- (4) Apply the previous step to  $V \setminus \phi(S)$  to prove that  $\phi^{-1}$  is continuous.

What happens if we drop the Hausdorff assumption?

**2.5. One point compactification.** A space is *locally compact* if every point has a compact neighborhood. There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. Namely, if  $X$  is locally compact Hausdorff, let  $\bar{X} = X \cup \{\infty\}$ . The open sets of  $\bar{X}$  are the open sets of  $X$  together with the sets  $(X \setminus K) \cup \{\infty\}$  where  $K$  is a compact subset of  $X$ . Prove that  $\bar{X}$  is a compact Hausdorff space and that the natural inclusion  $X \rightarrow \bar{X}$  is continuous and makes  $X$  into an open subset of  $\bar{X}$ . One calls  $\bar{X}$  the *one point compactification* of  $X$ . Discuss the one point compactification of  $\mathbb{R}^n$ .

**2.6. Proper maps.** A continuous map between topological spaces  $f : X \rightarrow Y$  is said to be *proper* if the inverse image of a compact subset of  $Y$  is a compact subset

of  $X$ . Prove that a proper map  $f$  of locally compact Hausdorff spaces extends uniquely to a continuous map between the one point compactifications  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ .

**2.7. Locally compact Hausdorff spaces.** Prove that for a Hausdorff space  $X$ , the following are equivalent:

- (1)  $X$  is locally compact
- (2)  $X$  is locally closed in a compact Hausdorff space
- (3)  $X$  is locally closed in a locally compact Hausdorff space

### 3. CONNECTEDNESS

**3.1. Connected closures.** Prove that the closure of a connected set is connected.

**3.2. Path connectedness.** A *path* in a topological space  $X$  is a continuous map  $f : [a, b] \rightarrow X$ . We say  $X$  is *path connected* if any two points  $x, y \in X$  can be joined by a path, i.e., if there is a path  $f : [a, b] \rightarrow X$  with  $f(a) = x$  and  $f(b) = y$ . Prove that a path connected space is connected. The converse is false; give a counterexample

**3.3. Connected components.** Let  $X$  be a topological space. Prove that the relation “ $x$  and  $y$  are contained in a connected subset of  $X$ ” is an equivalence relation. The equivalence classes for this relation are by definition the *connected components* of  $X$ . Prove that the components of  $X$  are closed in  $X$ . Give an example showing that they need not be open.

Prove something similar with “connected” replaced by “path connected.”

**3.4. Local connectedness.** A space  $X$  is said to be *locally connected* if every point has a connected neighborhood basis, i.e., if given  $p \in X$  and a neighborhood  $N$  of  $p$ , there is a connected neighborhood of  $p$  contained in  $N$ . Prove that the components of a locally connected space are open and closed (“clopen”).

Make a similar definition and prove a similar result for path connectedness.

**3.5. Identity component of a topological group.** Let  $G$  be a topological group. Prove that the connected component of  $G$  containing the identity element (the “identity component”) is a closed normal subgroup of  $G$ , call it  $G^0$ . Prove that  $G/G^0$  (with the quotient topology inherited from  $G$ ) is a topological group.