INTEGRATION WORKSHOP 2004
COMPLEX ANALYSIS EXERCISES

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1. Cauchy's Formula and Cauchy's Theorem

1. Suppose that γ is a piecewise smooth positively ("counterclockwise") oriented simple closed curve. Use Green's Theorem to show that the value of the integral

$$\int_\gamma \frac{dz}{z-p}$$

equals 0 if p is outside γ and $2\pi i$ if p is inside γ.

2. Show that there is no such function $f(z)$ defined in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, such that $f'(z) = \dfrac{1}{z}$.

3. Let γ be a piecewise smooth closed curve in $\mathbb{C}$ and p a point not on γ. Show that

$$w(\gamma, p) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-p}$$

is an integer number telling us how many times the curve γ "winds around" the point p in the counterclockwise direction. This number is called, quite appropriately, the winding number of γ around p.

4. Using the winding number show that the Cauchy's formulas for the function and its derivetives can be generalized to any closed curve and a point p not on the curve as follows:

$$f(p) \cdot w(\gamma, p) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-p}\, dz$$

and

$$f^{(n)}(p) \cdot w(\gamma, p) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{n+1}}\, dz.$$

5. If $f(z)$ is analytic in an neighbourhood of the disk $|z-p| \leq R$ and if $|f(z)| \leq M$ on the boundary of the disk, then

$$|f^{(n)}(p)| \leq \frac{n!}{R^n} M.$$

6. Setting $n = 1$ in the previous problem, prove the Liouville's Theorem: if $f(z)$ is a bounded entire function, then it must be constant.

7. Show that if there exists constants R, M such that for $|z| > R$ an entire function $f(z)$ satisfies $f(z) < M|z|^k$, then $f(z)$ is a polynomial of degree not exceeding k.

8. Use the Cauchy's Formula to deduce the Mean Value Formula:

$$f(p) = \frac{1}{2\pi}\int_0^{2\pi} f(p + re^{it})dt,$$

assuming that $f(z)$ is analytic inside and on the circle $z - p = re^{it}$.

9. Use the previous problem to show that if the absolute value of an analytic function $f(z)$ attains a relative maximum at $z = p$, then the function must be constant in some neighbourhood of p.

10. Prove the Maximum Modulus Principle. Let D be a connected open set bounded by a simple closed curve γ, and let $f(z)$ be a function analytic in D and on γ. Let M be the maximum value of $|f(z)|$ on γ. Show that $|f(z)| \leq M$ for all $z \in D$. Moreover, if the equality holds for a single point, then $f(z)$ is a constant function.

11. Prove the Schwarz Lemma. Let $f(z)$ be analytic in the open unit disk $|z| < 1$. And suppose $|f(z)| \leq 1$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover, if $|f(p)| = |p|$ for some $p \neq 0$ inside the disk, then $f(z) = cz$ for some constant c of absolute value one. *Hint:* Consider the function $g(z)$ such that $g(0) = f'(0)$ and $g(z) = f(z)/z$ for $z \neq 0$ and show that it is analytic. Then apply the maximum modulus principle.

2. The Residue Theorem and applications.

1. If $P(x)$ and $Q(x)$ are polynomials, such that $\deg(P) \leq \deg(Q) + 2$, and $Q(x)$ has no real roots, then the integral

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$$

can be computed by the Residue Theorem using the contour γ consisting of a semicircle $|z| = R$, $\mathrm{Im}(z) \geq 0$ together with the real line segment $[-R, R]$. Use this contour to compute the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 4x^2 + 5}\, dx.$$

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 6x + 10}\, dx$$

by using the same contour as in the previous problem.

3. Compute the integral

$$\int_{0}^{\infty} \frac{\log x}{(1 + x^2)(4 + x^2)}\, dx$$

by using the contour consisting of two semi-circles $|z| = R$, $\mathrm{Im}(z) \geq 0$ and $|z| = \varepsilon$, $\mathrm{Im}(z) \geq 0$, with $R > 2$ and $0 < \varepsilon < 1$, together with two real line segments $[-R, -\varepsilon]$ and $[\varepsilon, R]$.

4. Use the "keyhole" contour to compute the integral

$$\int_{0}^{\infty} \frac{\sqrt{x}}{x^2 + 5x + 4}.$$

3. Conformal maps.

1. *Linear Fractional Transformations.* A 2×2 matrix of complex numbers $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a map

$$z \mapsto \frac{az + b}{cz + d}$$

called the linear fractional transformation. Show that

- composition of two LFTs corresponds to product of matrices

- an LFT defines a continuous map from $\mathbb{C} \cup \{\infty\}$ to itself

- an LFT is constant if and only if the determinant $ad - bc$ is zero, otherwise it is bijective

- there is a unique LFT which maps a triple of distinct points (z_1, z_2, z_3) to $(0, 1, \infty)$ respectively

- write down an explicit formula for such LFT

- deduce that any triple of distinct points can be mapped by a unique LFT to any other triple of distinct points

- any LFT is a composition of three simple transformations: multiplication by a constant, adding a scalar and taking the reciprocal
- an LFT maps circles and lines to circles and lines
- which LFTs map the unit disk to itself?
- which LFTs map the upper-half plane to itself?
- find an LFT which maps a unit disk to the upper-half plane and find its inverse

2. Let us call an open subset U of $\mathbb{C}$ a domain of injectivity for an analytic function $f(z)$ if for two distinct $z_1, z_2 \in U$ we have $f(z_1) \neq f(z_2)$ and if for any larger open set containing U one can always find two distinct points with the same value. Find domains of injectivity for e^z, $\sin z$, $\cos z$, $\mathrm{Log}(z)$, and z^p, where $p \in \mathbb{Z}$.

3. Find a conformal map from the semi-infinite strip $0 < \mathrm{Re}(z) < 1$, $\mathrm{Im}(z) > 0$ to the upper-half plane.

4. Find a conformal map from the upper half-disk $|z| < 1$, $\mathrm{Im}(z) > 0$ to the full disk $|z| < 1$. *Hint*: first find a map to the first quadrant.

4. Harmonic functions.

1. Use the Cauchy-Riemann Equations to show that the real part of an analytic function is harmonic. Same for the imaginary part.

2. Two harmonic functions are called conjugate if they satisfy the Cauchy-Riemann Equations. Find a harmonic conjugate to those of the following functions which are harmonic:

$$\sqrt{x^2+y^2}, \quad \sin(x^2-y^2)\cosh(2xy), \quad x^4-6x^2y^2+y^4, \quad \frac{x}{x^2+y^2}, \quad \frac{\sqrt{x}}{x-y}.$$

3. Show that if a harmonic function attains a local maximum (or minimum), then it is locally constant.

4. Let r and θ be the polar coordinates for complex numbers, so $z = re^{i\theta}$. When is a function of r alone, $f(r)$, harmonic? Also, when is a function of θ alone, $g(\theta)$, harmonic?

5. Given a real-valued bounded continuous function $w(t)$, $t \in \mathbb{R}$, define another real-valued function in the upper-half plane:

$$f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y w(t)}{(x-t)^2 + y^2} \, dt.$$

Show that for a fixed x:

$$\lim_{y \to 0} f(x, y) = w(x).$$

6. Show that $f(x, y)$ from the previous problem is harmonic. Thus we have solved the Dirichlet problem for the Laplace equation in the upper-half plane, i.e. we found a harmonic function $f(x, y)$ with prescribed boundary value $f(t, 0) = w(t)$.