INTEGRATION WORKSHOP 2004
LINEAR ALGEBRA EXERCISES

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1. Vector spaces and linear transformations

You may assume that all vector spaces are finite dimensional.

1.1. A system of linear equations may have no solutions, a unique solution, or many solutions. Explain this in terms of the image and kernel of a linear transformation.

1.2. Let U_1 and U_2 be subspaces of a vector space V. Prove that $\dim U_1 + \dim U_2 = \dim(U_1 \cap U_2) + \dim(U_1 + U_2)$.

1.3. Let V and W be vector spaces and let $\operatorname{Hom}(V, W)$ be the set of linear transformations $V \to W$. Prove that $\operatorname{Hom}(V, W)$ has a natural vector space structure.

Choose bases $e_1, \ldots, e_m$ for V and $f_1, \ldots, f_n$ for W and let $\ell_1, \ldots, \ell_m$ be the dual basis for V^*. For $1 \leq i \leq n$ and $1 \leq j \leq m$ prove that the formula

$$E_{ij}(v) = \ell_j(v) f_i$$

defines an element $E_{ij} \in \operatorname{Hom}(V, W)$.

Prove that the E_{ij} form a basis of $\operatorname{Hom}(V, W)$ and conclude that $\dim \operatorname{Hom}(V, W) = (\dim V)(\dim W)$. If you know about tensor products, this proof allows you to show that $\operatorname{Hom}(V, W) \cong V^* \otimes W$.

To what extent does the above work if we drop the finite dimensional hypothesis?

1.4. An element $T \in \operatorname{Hom}(V, V) = \operatorname{End}(V)$ is called *nilpotent* if $T^n = 0$ for some n. Prove that T is nilpotent if and only if there exists a basis of V in which the matrix of T is strictly upper triangular, i.e., upper triangular with zeroes on the diagonal. Conclude that if $T^n = 0$ for some n, then $T^{\dim V} = 0$.

1.5. An element $T \in \operatorname{Hom}(V, V) = \operatorname{End}(V)$ is called *semi-simple* if there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that $V \cong \oplus V_i$ and $T(V_i) \subset V_i$ for all i. (This is not the standard definition unless the ground field is algebraically closed, but we'll ignore this for now.) Prove that V is semi-simple if and only if there exists a basis of V in which the matrix of T is diagonal.

1.6. Let T be any element of $\operatorname{Hom}(V, V) = \operatorname{End}(V)$. Prove that there exists a basis of V in which the matrix of T is upper triangular if and only if there exist subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$ and $T(V_i) \subset V_i$.

Recall or prove that if the ground field is $\mathbb{C}$ then T has an eigenvector. (Hint: Let v be any non-zero vector and consider the $n+1$ vectors $v, TV, T^2v, \ldots, T^n v$. They must be linearly dependent, so ...)

Prove by induction on the dimension of V that the equivalent conditions in the first paragraph are always satisfied if the ground field is $\mathbb{C}$.

1.7. If $T : V \to W$ is a linear transformation and $T^* : W^* \to V^*$ is its transpose, prove that the kernel of T is the orthogonal complement of the image of T^*, in other words,

$$\ker T = \{v \in V \mid \ell(v) = 0 \text{ for all } \ell \in \operatorname{Im} T^*\}.$$

Similarly, the image of T is the orthogonal complement of the kernel of T^*.

1.8. Prove that if $W \subset V$ is a subspace with $\dim W = \dim V - 1$ ("codimension 1") then there exists an element $\ell \in V^*$ such that W is the kernel of ℓ. Generalize to an arbitrary subspace $W \subset V$.

1.9. Suppose that the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, \dots, m$$

has real coefficients: $a_{ij}, b_i \in \mathbb{R}$. Prove that if the system has a solution with $(x_j) \in \mathbb{C}^n$, then it has a solution with $(x_j) \in \mathbb{R}^n$. In this case, is every solution in $\mathbb{R}^n$?

1.10. Let V be the vector space of polynomials of degree $\leq d$. Assuming $d \geq 3$, prove in two different ways that the set W of polynomials divisible by $(x-2)^2(x-3)$ has dimension $d - 2$.

1.11. Convince yourself that Gaussian elimination leads to an algorithm for writing an arbitrary matrix as a product LPU where L is lower triangular, P is a permutation matrix, and U is upper triangular.

2. Endomorphisms and the Jordan form

In this section we assume that all vector spaces are finite dimensional over $\mathbb{C}$.

2.1. (Uniqueness of the abstract Jordan decomposition) The abstract Jordan decomposition says that every $T \in \operatorname{End}(V)$ can be written $T = S + N$ where S is semi-simple, N is nilpotent, and $SN = NS$. The proof also shows that there are polynomials p and q such that $S = p(T)$ and $N = q(T)$.

Suppose that $T = S' + N$" where S' is semi-simple, N' is nilpotent, and $S'N' = N'S'$. Prove that $SS' = S'S$ and $NN' = N'N$.

Prove that $S - S'$ is semi-simple and $N - N'$ is nilpotent. (These statements would be false in general if S and S' did not commute or N and N' did not commute.)

Conclude that $S = S'$ and $N = N'$.

2.2. Give a recipe for the sizes of the Jordan blocks of an endomorphism $T : V \to V$ in terms of the dimensions of $\ker(T - \lambda)^i$ where λ is an eigenvalue of T and $i = 1, 2, \dots$.

2.3. Give necessary and sufficient conditions for an endomorphism to have a square root. I.e., given $T : V \to V$, when is there an endomorphism U such that $U^2 = T$?

2.4. Prove that every $n \times n$ complex matrix is conjugate to its transpose.

2.5. Prove that the set of diagonalizable $n \times n$ matrices is dense in $\mathbb{C}^{n^2}$, in the following sense: Given A, for every $\epsilon > 0$ there is a matrix B all of whose entries are of absolute value $< \epsilon$ such that $A + B$ is diagonalizable.

2.6. Prove that if $T^m = Id$ for some $m \geq 1$ then T is diagonalizable.

2.7. Let $T : V \to V$ be an endomorphism. If $v \in V$, the *cyclic subspace* generated by v is the span of v, Tv, T^2v, ... Use the Jordan form of T to determine the possible dimensions of cyclic subspaces (for various $v \in V$), and the possible Jordan forms for T restricted to a cyclic subspace.

2.8. Determine the structure of the set of matrices commuting with a given matrix. You may want to consider several special cases first: diagonal matrices with distinct eigenvalues, diagonal matrices with a repeated eigenvalue, a single Jordan block, ...

3. Bilinear forms

3.1. Use the Gram-Schmidt process to prove that every invertible complex matrix can be written as the product of a unitary matrix and an upper triangular matrix. ("QR decomposition") What is the real version?

3.2. Prove that an invertible real matrix A can be written as PQ where P is symmetric positive definite and Q is orthogonal. ("polar decomposition") Hint: Consider AA^t, which is a symmetric positive definite matrix, and take its square root. What is the complex version?

3.3. Suppose $(,) : V \times W \to \mathbb{C}$ is a bilinear form. Define the left kernel V_0 by

$$V_0 = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$$

and similarly for the left kernel W_0. Prove that $(,)$ induces a non-degenerate bilinear form on $(V/V_0) \times (W/W_0)$ and in particular, $\dim(V/V_0) = \dim(W/W_0)$.

3.4. A *quadratic form* on a complex vector space V is a function $q : V \to \mathbb{C}$ such that $q(av) = a^2q(v)$ for all $v \in V$, $a \in \mathbb{C}$ and such that

$$(v, w) = \frac{1}{2}\left(q(v + w) - q(v) - q(w)\right)$$

is bilinear. If $(,)$ is a bilinear form on V, prove that $q(v) = (v, v)$ is a quadratic form on V and that every quadratic form on V arises in this way. Do q and $(,)$ uniquely determine each other?

3.5. Let P_d be the space of polynomials of degree $\leq d$. Compute the signature of the symmetric bilinear form $(f, g) = (fg)^{(d)}(0)$. (The right hand side is the d-th derivative of the product fg evaluated at zero.)

3.6. Let P_2 be the space of real polynomials of degree ≤ 2 equipped with the Euclidean form $(f, g) = \int_0^1 fg$. Find an orthonormal basis of P_2 consisting of monic polynomials.

3.7. The following statement is false: if $T : V \to V$ is an endomorphism of a finite dimensional complex vector space, then there exists a positive definite Hermitan inner product on V with respect to which T is normal. Explain why and then add a hypothesis to make the conclusion true. What happens if "normal" is changed to "unitary" or "hermitian"?

3.8. Prove that the set of unitary $n \times n$ matrices is a compact subset of $\mathbb{C}^{n^2}$.