1 Linear Algebra Problems

1. Let A^* be the conjugate transpose of the complex matrix A, i.e., $A^* = \left(\bar{A}\right)^t$. A is said to be Hermitian if $A^* = A$, real symmetric if A is real and $A^t = A$, skew-Hermitian if $A^* = -A$ and normal if $A^*A = AA^*$.

Find the dimension and a basis for each of the following vector spaces.
(a) $M_n(\mathbb{C})$, $n \times n$ complex matrices, over $\mathbb{C}$.
(b) $M_n(\mathbb{C})$ over $\mathbb{R}$
(c) $H_n(\mathbb{C})$, $n \times n$ Hermitian matrices, over $\mathbb{R}$
(d) $H_n(\mathbb{R})$, $n \times n$ real symmetric matrices, over $\mathbb{R}$
(e) $S_n(\mathbb{C})$, $n \times n$ skew-Hermitian matrices, over $\mathbb{R}$
(f) The space consisting of all real polynomials of A over $\mathbb{R}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \ \omega = \frac{-1+\sqrt{3}i}{2}$$

Is $H_n(\mathbb{C})$ a vector space over $\mathbb{C}$? IS the set of $n \times n$ normal matrices a subspace of $M_n(\mathbb{C})$? Show that $M_n(\mathbb{C}) = H_n(\mathbb{C}) + S_n(\mathbb{C})$, i.e., any $n \times n$ matrix is a sum of Hermitian matrix and a skew-Hermitian matrix.

2. Find the space of matrices commuting with
(a) $A = I_n$
(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
(c) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a \neq b$
(d) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(e) All $n \times n$ matrices

3. True or False. If true, what is the dimension? Basis?
(a) $\left\{(x, y) : x^2 + y^2 = 0, x, y \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$.
(b) $\left\{(x, y) : x^2 + y^2 = 0, x, y \in \mathbb{C}\right\}$ is a subspace of $\mathbb{C}^2$.
(c) $\left\{(x, y) : x^2 - y^2 = 0, x, y \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^2$.
(d) $\{(x, y) : x - y = 0, x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.
(e) $\{(x, y) : x - y = 1, x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.
(f) $\{p(x) : p(x) \in \mathbb{P}[x] \text{ has degree } 3\}$ is a subspace of $\mathbb{P}[x]$.
(g) $\{p(x) : p(0) = 0, p(x) \in \mathbb{P}[x]\}$ is a subspace of $\mathbb{P}[x]$
(h) $\{p(x) : 2p(0) = p(1)\}$ is a subspace of $\mathbb{P}[x]$.
(i) $\{p(x) : p(x) \geq 0, p(x) \in \mathbb{P}[x]\}$ is a subspace of $\mathbb{P}[x]$.

4. Show that $M_2(\mathbb{R}) = W_1 \oplus W_2$, where

$$W_1 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} c & d \\ d & -c \end{pmatrix} : c, d \in \mathbb{R} \right\}.$$

5. Let A be an $n \times n$ real matrix.
(a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.
(b) Show that if $A^2 + I = 0$, then n must be even.
(c) Does (b) remain true for complex matrices?

6. Introduce the correspondence between complex numbers and real matrices:

$$z = x + iy \,\tilde{}\, Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in M_2(\mathbb{R}),$$

and define for each pair of complex numbers u and v :

$$q = (u, v) \cong C(q) = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in M_2(\mathbb{C})$$

(a) Show that $\bar{z} \,\tilde{}\, Z^t$.
(b) Show that $ZW = WZ$
(c) Show that $z \,\tilde{}\, Z$ and $w \,\tilde{}\, W$ imply $zw + ZW$.
(d) Find Z^n, where $z = r(\cos\theta + i \sin\theta)$.
(e) What is the matrix corresponding to i?
(f) Show that $|C(q)| \geq 0$. Find $C(q)^{-1}$ when $|u|^2 + |v|^2 = 1$.
(g) Replace each entry of $C(q)$ with the corresponding 2×2 real matrix to the entry to get

$$\mathcal{R}(q) = \begin{pmatrix} U & V \\ -V^t & U^t \end{pmatrix} \in M_4(.)$$

Then $|\mathcal{R}(q)| \geq 0$.
(h) Show that $\mathcal{R}(q)$ is similar to a matrix of form

$$\begin{pmatrix} U & X \\ -X & U \end{pmatrix}$$

for some X.
(i) Show that $\mathcal{R}(q)$ is singular if and only if $\mathcal{C}(q)$ is singular if and only if $u = v = 0$.

7. True or false
(a) For any $m \times n$ matrix A with rank r, there exists invertible $m \times m$ and $n \times n$ matrices P and Q such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q.$$

(b) For any $n \times n$ matrix A with rank r, there exists an invertible $n \times n$ matrix P such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

(c) If A is a real matrix and A^{-1} exists over $\mathbb{C}$, then A^{-1} is also a real matrix.
(d) If $(A^*)^2 = A^2$, then $A^* = A$ or $A^* = -A$.
(e) If $\operatorname{rank} A = \operatorname{rank} B$, then $\operatorname{rank}(A^2) = \operatorname{rank}(B^2)$
(f) $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$

(g) $\operatorname{rank}(A-B) \leq \operatorname{rank}(A) - \operatorname{rank}(B)$

(h) Since $(1, i)$ and $(i, -1)$ are linearly independent over $\mathbb{R}$, the matrix $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is invertible.

8. Suppose that A and B are both $m \times n$ matrices. Show that $Ax = 0$ and $Bx = 0$ have the same solution space if and only if there exists an invertible matrix C such that $A = CB$. Use this fact to show that if $\operatorname{rank}\left(A^2\right) = \operatorname{rank}(A)$, then there exists an invertible matrix D such that $A^2 = DA$.

9. What are the matrices that are similar to themselves only?

10. Prove assertions (a) and (b) and construct an example for (c).

(a) Let $A \in M_n(\mathbb{C})$. If the eigenvalues of A are distinct from each other, then A is diagonalizable, i.e., there is an invertible matrix P such that $P^{-1}AP$ is diagonal.

(b) If matrix A commutes with a matrix with distinct eigenvalues, then A is diagonalizable.

(c) Give an example of a matrix A that is diagonalizable but not unitary diagonalizable, that is, $P^{-1}AP$ is diagonal for some invertible P, but U^*AU is not diagonal for any unitary matrix U.

11. True or false

(a) If $A^k = 0$ for all positive integers $k \geq 2$, then $A = 0$.

(b) If $A^k = 0$ for some integer k, then $\operatorname{tr} A = 0$.

(c) If $\operatorname{tr} A = 0$, then $|A| = 0$.

(d) If A and B are similar, then $|A| = |B|$

(e) If A and B are similar, then they have the same eigenvalues.

(f) If A and B have the same eigenvalues, then they are similar.

(g) If A and B have the same characteristic polynomial, then they have the same eigenvalues.

(h) If A and B have the same eigenvalues, then they have the same characteristic polynomial.

(i) If A and B have the same characteristic polynomial, then they are similar.

(j) If $\operatorname{tr} A^k = \operatorname{tr} B^k$ for all positive integers k, then $A = B$.

(k) If the eigenvalues of A are $\lambda_1, ..., \lambda_n$, then A is similar to the diagonal matrix $diag\{\lambda_1, ..., \lambda_n\}$.

(l) $diag\{1, 2, ..., n\}$ is similar to $diag\{n, ..., 2, 1\}$

(m) If A has a repeated eigenvalue, then A is not diagonalizable.

(n) If A is diagonalizable, then A is normal.

(o) If A is unitarily diagonalizable, then A is normal.

(p) If A has r nonzero eigenvalues, then $\operatorname{rank}(A) \geq r$.

12. Let $A \in M_n(\mathbb{C})$ and $A \neq 0$. Define a transformation on $M_n(\mathbb{C})$ by

$$\mathcal{T}(X) = AX - XA, \ X \in M_n(\mathbb{C})$$

Show that

(a) $\mathcal{T}$ is linear
(b) Zero is an eigenvalue of T
(c) If $A^k = 0$, then $\mathcal{T}^{2k} = 0$.
(d) If A is diagonalizable, so is $\mathcal{T}$.
(e) $\mathcal{T}(XY) = X\mathcal{T}(Y) + \mathcal{T}(X)Y$.
(f) If A and B commute, so do $\mathcal{T}$ and $\mathcal{L}$ where $\mathcal{L}$ is defined as
$\mathcal{L}(X) = BX - XB,\ X \in M_n(\mathbb{C})$.
Find all A such that $\mathcal{T}$ =0 and discuss the converse of (f).

13. Let W be an invariant subspace of a linear transformation $\mathcal{A}$ on a finite-dimensional vector space V.
(a) Show that if $\mathcal{A}$ is invertible, then W is also invariant under $\mathcal{A}^{-1}$.
(b) If $V = W \oplus W'$, is W' necessarily invariant under A?

14. Show that if A is an invertible Hermitian matrix, then there exists and invertible matrix P such that $P^*AP = A^{-1}$.

15. Is it possible for some non-Hermitian matrix $A \in M_n(\mathbb{C})$ to satisfy $x^*Ax \geq 0$ for all $x \in \mathbb{R}^n$? $x \in \mathbb{C}^n$?

16. Construct examples
(a) Matrices A and B that have only positive eigenvalues, AB has only negative eigenvalues. (Note that A and B are not necessarily Hermitian).
(b) Is it possible that $A + B$ has only negative eigenvalues for matrices A and B with positive eigenvalues?
(c) Matrices A, B, and C are positive definite, ABC has only negative entries.
(d) Is it possible that the matrices in (c) are 3×3?

17. Let $A \in M_n(\mathbb{C})$ be a normal matrix. Show that
(a) $\ker A^* = \ker A$
(b) $\operatorname{Im} A^* = \operatorname{Im} A$
(c) $\mathbb{C}^n = \operatorname{Im} A \oplus \ker A$

18. A permutation matrix is a matrix which has exactly one 1 in each row and each column.
(a) How many $n \times n$ permutation matrices are there?
(b) The product of two permutation matrices of the same size is also a permutation matrix. How about the sum?
(c) Show that any permutation matrix is invertible and its inverse is equal to its transpose.
(d) For what permutation matrices P, does $P^2 = I$?

19. Let P be the $n \times n$ permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}$$

Then show the following

(a) For any positive integer $k \leq n$,

$P^k = \begin{pmatrix} 0 & I_{n-k} \\ I_k & 0 \end{pmatrix}$, and

$P^{n-1} = P^t$, $P^n = I_n$

(b) $P, P^2, ..., P^n$ are linearly independent.

(c) $P^i + P^j$ is a normal matrix

(d) For $n \geq 3$, P is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$.

(e) For every P^i, there exists a permutation matrix T such that $T^{-1}P^iT = P$.

20. Let $\mathcal{A}$ be a linear transformation on an inner product space V. Show that for any unit vector $x \in V$

$(Ax, x)(x, Ax) \leq (Ax, Ax)$.

In particular, for $A \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$ with $||x|| = 1$,

$x^*A^*xx^*Ax \leq x^*A^*Ax$.

21. Let $e_1, e_2, ..., e_n$ be vectors of an inner product space over a field $\mathbb{F}$, and let $A = (a_{ij})$, where

$a_{ij} = (e_i, e_j)$, $i, j = 1, 2, ..., n$.

Show that $e_1, e_2, ..., e_n$ are linearly independent if and only if A is nonsingular.

22. Let V be an inner product space over $\mathbb{R}$.

(a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$(e_i, e_j) < 0$, $i, j = 1, 2, 3$, $i \neq j$,

show that e_1, e_2, e_3 are linearly independent.

(b) Is it possible for three vectors in the xy-plane to have pairwise negative products?

(c) Does (a) remain valid when the word "negative" is replaced with "positive"?

(d) Suppose that u, v, and w are three unit vectors in the xy-plane. What are the maximum and minimum values that

$(u, v) + (v, w) + (w, u)$

can attain? and when?

23. If $\{e_1, ..., e_n\}$ is an orthonormal basis for an inner product space V over $\mathbb{C}$, and $x \in V$, show that

$x = \sum_{i=1}^{n} (x, e_i) e_i$

and

$(x, x) \geq \sum_{i=1}^{k} |(x, e_i)|^2$, $1 \leq k \leq n$.

Why are pairwise orthogonal nonzero vectors linearly independent?

24. Let W be a subspace of an inner product space V and let S be a subset of V. Answer true or false.

(a) There is a unique subspace W' such that $W' + W = V$.

(b) There is a unique subspace W' such that $W' \oplus W = V$.

(c) There is a unique subspace W' such that $W' \oplus W = V$ and $(w, w') = 0$ for all $w \in W$ and $w' \in W'$/

(d) $\left(W^{\perp}\right)^{\perp} = W$.

(e) $\left(S^{\perp}\right)^{\perp} = S$

(f) $\left[\left(S^{\perp}\right)^{\perp}\right]^{\perp} = S^{\perp}$

(g) $\left(S^{\perp} + W\right)^{\perp} = \left(S^{\perp}\right)^{\perp} \cap W^{\perp}$

(h) $\left(S^{\perp} \cap W\right)^{\perp} = \left(S^{\perp}\right)^{\perp} + W^{\perp}$