# Integration Workshop

## Analysis Notes

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## CALCULUS

- **Multivariable differential calculus**

### - Total derivative

**Definition.** Let $S \subset R^n$ and $f : S \to R^m$. Let $c \in \text{int}S$ and choose $\epsilon$ so that $B(c, \epsilon) \subset S$. The function $f$ is said to be *differentiable* at $c$ if there exists a linear function, the *total derivative*,

$$T_c^f : R^n \to R^m \quad \text{such that} \quad f(c+v) = f(c) + T_c^f(v) + |v| E_c(v),$$

for $|v| < \epsilon$ where $E_c(v) \to 0$ as $v \to 0$.

If this holds for every $c \in S$, we say that $f$ is *differentiable.* For $m = 1$, $T_c^f(v) = \nabla f(c) \cdot v$.

### - Jacobian matrix

If the total derivative exists then the directional derivatives

$$D_v f(c) = \lim_{\epsilon \to 0} \frac{f(c + \epsilon v) - f(c)}{\epsilon}$$

exist and equal $T_c^f(v)$.

Let $e_1, \ldots, e_n$ be the standard basis for $R^n$, then the *partial derivatives* are denoted

$$\frac{\partial}{\partial x_k} = D_{e_k}$$

and

$$T_c^f(e_k) = \frac{\partial}{\partial x_k} f(c) = (\frac{\partial}{\partial x_k} f_1(c), \ldots, \frac{\partial}{\partial x_k} f_m(c)).$$

The matrix representation of $T$ in this basis is called the *Jacobian matrix*

$$Df(c) = \begin{pmatrix} \frac{\partial}{\partial x_1} f_1(c) & \frac{\partial}{\partial x_2} f_1(c) & \cdots & \frac{\partial}{\partial x_n} f_1(c) \\ \frac{\partial}{\partial x_1} f_2(c) & \frac{\partial}{\partial x_2} f_2(c) & \cdots & \frac{\partial}{\partial x_n} f_2(c) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(c) & \frac{\partial}{\partial x_2} f_m(c) & \cdots & \frac{\partial}{\partial x_n} f_m(c) \end{pmatrix}.$$

### - Chain rule

Suppose that $a \in R^p, b = g(a) \in R^n$ and $f(b) \in R^m$. Write $h = f \circ g$, then the chain rule state that

$$T_a^h = T_b^f \circ T_a^g.$$

For the Jacobian matrices, note that

$$Dh(a) \text{ is an } m \times p \text{ matrix}, \quad Df(b) \text{ is an } m \times n \text{ matrix}, \quad Dg(a) \text{ is an } n \times p \text{ matrix}.$$

Because the matrix of a composition is the product of the corresponding matrices, the matrix form of the chain rule states that

$$Dh(a) = Df(b)Dg(a).$$

### - Mean value theorem

Let $L(x_1, x_2) = \{\lambda x_1 + (1 - \lambda)x_2 : 0 \leq \lambda \leq 1\}$ be the line segment connecting $x_1$ and $x_2$ in $R^n$.

**Mean Value Theorem.** Let $S$ be an open subset of $R^n$ and assume that $f : S \to R^m$ is differentiable at each point of $S$. Choose $x_1$ and $x_2$ so that $L(x_1, x_2) \subset S$. Then for every vector $a \in R^m$, there is a point $c \in L(x_1, x_2)$ such that

$$a \cdot (f(x_2) - f(x_1)) = a \cdot T_c^f(x_2 - x_1).$$

### - Higher order derivatives

**Theorem** Let $f : R^n \to R^m$, then the following conditions are sufficient for the equality of the mixed partial derivatives

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(c) = \frac{\partial^2 f}{\partial x_j \partial x_i}(c) \qquad i, j = 1, \ldots, n.$$

1. Both $\partial f / \partial x_i$ and $\partial f / \partial x_j$ exist in an $n$-ball $B(c; \delta)$ and are differentiable at $c$.
2. Both $\partial f / \partial x_i$ and $\partial f / \partial x_j$ exist in an $n$-ball $B(c; \delta)$ and $\partial^2 f / \partial x_i \partial x_j$ and $\partial^2 f / \partial x_j \partial x_i$ are both continuous at $c$.

Call $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index if each of its entries are non-negative integers. Write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. This allows for the notational abbreviations

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha^n} \quad \text{and} \quad D_\alpha = \frac{\partial^{\alpha^1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha^n}}{\partial x_n^{\alpha_n}}.$$

and provides for a compact notation for *Taylor's formula* for functions $f$ from $R^n$ to $R^1$. Write $f^{(k)}(x; t) = \sum_{|\alpha|=k} D_\alpha f(x) t^\alpha$. and assume that $f$ and all of its partial derivatives of order up to $m - 1$ are differentiable at each point of an open set $S \subset R^n$. Choose $x$ and $a$ so that $L(a, x) \subset S$ then for some $c \in L(a, x)$

$$f(x) = f(a) + \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(x; t) + \frac{1}{m!} f^{(m)}(c; t).$$

## • Implicit functions and extremum problems

Let $A$ be a $n \times n$ matrix. Then, for $x, t \in R^n$,

$$Ax = y$$

has a unique solution whenever $A$ has nonzero determinant. This suggests in looking for a unique solution to $f(x) = y$, we consider the *Jacobian determinant*, the determinant of the Jacobian matrix,

$$J_f(x) = \det Df(x) = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

### - Inverse function theorem

**Theorem.** Let $f : S \to R^n$ be continuously differentiable on $S \subset R^n$. If the Jacobian determinant $J_f(a) \neq 0$ for some point $a \in S$, then there exists two open sets $X \subset S$ and $Y \subset f(S)$ and a uniquely determined function $g$ defined on $Y$ such that

1. $a \in X$ and $f(a) \in Y$,
2. $Y = f(X)$,
3. $f$ is one-to-one on $X$,
4. $g(Y) = X$,
5. $g(f(x)) = x$ for every $x \in X$, and
6. $g$ is continuously differentiable on $Y$.

Note that for $y = f(x)$, $Dg(y) \cdot Df(x)$ is the identity matrix.

### - Implicit function theorem

**Theorem.** Let $S \subset R^n \times R^k$ and suppose that $f : S \to R^n$ is continously differentiable. Assume that $f(x_0, y_0) = 0$ and that the $n \times n$ determinant det $[\partial f_j / \partial x_i(x_0, y_0)] \neq 0$. Then there exists a $k$-dimensional set $Y_0$ containing $y_0$ and a unique vector valued function $g : Y_0 \to R^n$ such that

1. $g$ is continuously differentiable,
2. $g(y_0) = x_0$, and
3. $f(g(y), y) = 0$ for every $y \in Y_0$.

### - Extremum problems

If $f : S \to R^1$ is differentiable at $a$ and $\nabla f(a) = 0$, then $a$ is called a *stationary point* of $f$. A stationary point $a$ is called a *saddle point* if every $n$-ball about $a$ contains points $x_-$ and $x_+$ so that

$$f(x_-) < f(a) < f(x_+).$$

The *second derivative test* applies whenever all the second-order partial derivatives exist and are continuous in an $n$-ball about $a$. Consider the quadratic form

$$Q(t) = \frac{1}{2} f''(a; t) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(a) t_i t_j.$$

Then

1. If $Q(t) > 0$, for all $t \neq 0$, then $f$ has a relative minimum at $a$.
2. If $Q(t) < 0$, for all $t \neq 0$, then $f$ has a relative maximum at $a$.
3. If $Q(t)$ takes on positive and negative values, then $f$ has a saddle point at $a$.
4. In other cases the test is inconclusive.

### - Lagrange multipliers

One extremum problem may be to maximize $f(x, y)$ subject to the constraint that $y = g(z)$. However, substituing may be impractical if the functional relationship between $y$ and $z$ is only known implicitly. Lagrange's method provides a useful necessary condition to proceed.

**Theorem.** Let $S$ be an open subset of $R^n$ and let $f, g_1, \ldots, g_m$, $m < n$ be real valued continuously differentiable functions on $S$. Write $g = (g_1, \ldots, g_m)$ and

$$G_0 = \{x : g(x) = 0\}.$$

Assume

- there exists $x_0 \in G_0$ and an $n$-ball $B(x_0)$ such that either
  - $f(x) \leq f(x_0)$ for all $x \in G_0 \cap B(x_0)$ or
  - $f(x) \geq f(x_0)$ for all $x \in G_0 \cap B(x_0)$.
- det $[\partial g_i / \partial x_j(x_0)] \neq 0$.

Then there exists $\lambda_1, \ldots, \lambda_m$ such that

$$\nabla f(x_0) + \lambda_1 \nabla g_1(x_0) + \cdots + \lambda_m \nabla g_m(x_0) = 0.$$

The functions $g_1, \ldots, g_m$ are called the *side conditions.* The numbers $\lambda_1, \ldots, \lambda_m$ are known as *Lagrange multipliers.*

## • Multivariable Riemann integral

### - Evaluation of a multiple integral

Define a $k$-cell by $I_k = [a_1, b_1] \times \cdots \times [a_k, b_k]$. For $f : I_k \to R$, define

$$\int_{I_k} f(x)\ dx = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} f(x_1, \ldots, x_k) dx_k \cdots dx_2 \right) dx_1.$$

The value of this integral is the limit of the value of Riemann sums $S(P, f) = \sum_{i_1} \cdots \sum_{i_k} f(t_{i_1 \cdots i_k})\, \Delta x_1 \cdots \Delta x_k$. It is evaluated above as interated one-dimensional Riemann integrals. The value of the integral does not depend on the order that the iterated integrals are performed.

### - Transformation formulas

Let $T$ be a one-to-one continuously differentiable mapping of an open set $V \in R^k$ into $R^k$ such that the Jacobian determinant $J_T(x) \neq 0$ for all $x \in E$. Let $f$ be a continuous function on $R^k$ whose support is compact and lies in $T(V)$, then

$$\int_{R^k} f(y)\ dy = \int_{R^k} f(T(x))|J_T(x)|\ dx.$$

### - Differential forms

Let $K \subset R^k$ be compact and let $V \subset R^n$ be open A *k-surface* is a continuously differentiable mapping $\Phi : K \to V$. For example, each component of a 1-surface is called a *curve.*

A *differential form of order k*, or briefly, a *k-form*, is a function $\omega$, represented symbolically by

$$\omega = \sum a_{i_1 \cdots i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

that assigns to each $k$-surface $\Phi$ in $V$ a number

$$\int_\Phi \omega = \int_K \sum a_{i_1 \cdots i_k}(\Phi(u)) \frac{\partial(x_{i_1}, \ldots x_{i_k})}{\partial(u_1, \ldots u_k)}\ du.$$

A 0-form is defined to be a continuous function of $V$. Integrals of 1-forms are called *line integrals.*

Let $c \in R$ and let $\omega, \omega_1, \omega_2$ be $k$-forms on $E$, then:

- $\int_\Phi c\omega = c \int_\Phi \omega$.
- $\int_\Phi (\omega_1 + \omega_2) = \int_\Phi \omega_1 + \int_\Phi \omega_2$.
- For $\omega = a_{i_1 \cdots i_k}(x)\ dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and for $\bar{\omega}$ obtained from $\omega$ by interchanging some pair of subscripts, $\bar{\omega} = -\omega$.

Write the basic $k$-form $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, for $1 \leq i_1 < \cdots < i_k$, giving the standard presentation

$$\omega = \sum_I a_I(x) dx_I$$

.

**- Differentiation**

The operator $d$ is a mapping from $k$-forms to $(k+1)$-forms defined as follows:

1. For a class $C^1$ 0-form, $df = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(x) dx_i$.
2. For the class $C^1$ $k$-form $\omega$ above in the standard presentation, $d\omega = \sum_I (da_I(x)) \wedge dx_I$.

For $i = 1, 2$, let $\omega_i$ be class $C^1$ $k_i$-form, then $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d\omega_2$. If $\omega$ is of class $C^2$, $d(d\omega) = 0$.

**Definition.**

1. A $k$-form $\omega$ is called *exact* if $\omega = d\zeta$ for some $(k-1)$-form $\zeta$.
2. A class $C^1$ $k$-form is called *closed* if $d\omega = 0$.

Every exact class $C^1$ form is closed. If the domain is a convex set, then the *Poincaré lemma* states that the converse is true.

**- Stokes' theorem**

**Stokes' Theorem.** If $\Psi$ is a $k$-chain of class $C^2$ in an open set $V \subset R^m$ and if $\omega$ is a $(k-1)$-form of class $C^1$ in $V$, then

$$\int_\Psi d\omega = \int_{\partial\Psi} \omega.$$

Relating this to the theorems in multivariable calculus:

| $k$ | $m$ | theorem |
|---|---|---|
| 1 | 1 | fundamental theorem |
| 2 | 2 | Green's theorem |
| 3 | 3 | divergence theorem |
| 2 | 3 | classical Stokes'theorem |

**The Divergence Theorem.** Let $F$ be a continuously differentiable vector field on an open set $V \subset R^3$ and let $C \subset V$ be closed with positively oriented boundary $\partial V$, then

$$\int_C (\nabla \cdot F) \; dV = \int_{\partial C} (F \cdot \hat{n}) \; dA$$

where $\hat{n}$ is an outward unit normal.

**Stokes' Formula.** Let $F$ be a continuously differentiable vector field on an open set $V \subset R^3$ and let $S \subset V$ be a 2-surface of class $C^2$, then

$$\int_S (\nabla \times F) \cdot \hat{n} \; dA = \int_{\partial S} (F \cdot \hat{t}) \; ds.$$

where $\hat{t}$ is an oriented unit tangent vector.

# REAL ANALYSIS

## Sequences and series

### - Monotone sequences

By the completeness axiom of the real numbers, a monotone sequence converges if and only if it is bounded. Let $\{a_n : n \geq 1\}$ and define $b_n = \sup\{a_k : k \geq n\}$ then $\{b_n : n \geq 1\}$ is a nonincreasing sequence and so has a limit, call this the *limit superior* or *upper limit* of $\{a_n : n \geq 1\}$ and write

$$\limsup_{n\to\infty} a_n.$$

Similarly, define $c_n = \inf\{a_k : k \geq n\}$ then $\{c_n : n \geq 1\}$ is a nondecreasing sequence, call its limit *limit inferior* or *lower limit* of $\{a_n : n \geq 1\}$ and write

$$\liminf_{n\to\infty} a_n.$$

### - Convergence tests for series and sequences

1. **Integral test.** Let $f$ be a positive decreasing function defined on $[1, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$. For $n = 1, 2, \ldots$ define

$$s_n = \sum_{k=1}^{n} f(k), \qquad t_n = \int_1^n f(x)\, dx, \qquad d_n = s_n - t_n.$$

   Then

   (a) $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$, for $n = 1, 2, \ldots$.

   (b) $d = \lim_{n\to\infty} d_n$ exists.

   (c) $\{s_n : n \geq 1\}$ converges if and only if $\{t_n : n \geq 1\}$ converges

   (d) $0 \leq d_k - d \leq f(k)$, for $k = 1, 2, \ldots$.

2. **Ratio and root tests.** Given a series $\sum_{n=1}^{\infty} a_n$ of nonzero complex terms, let

$$r_- = \liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}|, \qquad r_+ = \limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}|, \qquad \rho = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

   (a) The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if either $r_+ < 1$ or $\rho < 1$.

   (b) The series $\sum_{n=1}^{\infty} a_n$ diverges if either $r_- > 1$ or $\rho > 1$.

   (c) The tests are inconclusive if $r_- \leq 1 \leq r_+$ and $\rho = 1$.

3. **Dirichlet's test.** Given a series $\sum_{n=1}^{\infty} a_n$ of nonzero complex terms whose partial sums form a bounded sequence. Let $\{b_n : n \geq 0\}$ be a decreasing sequence converging to 0, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

4. **Abel's test.** The series $\sum_{n=1}^{\infty} a_n b_n$ converges if $\sum_{n=1}^{\infty} a_n$ converges and if $\{b_n : n \geq 0\}$ is monotone and bounded.

**- Rearrangement of series**

Let $k : Z^+ \to Z^+$ be a bijection then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are called *rearrangements* if

$$b_n = a_{k(n)}.$$

- If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent with sum $s$, then every rearrangement of also converges absolutely with sum $s$.
- Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series of real valued terms and choose $x_-, x_+ \in [-\infty, +\infty]$, $x_- \leq x_+$. Then there exists a rearangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that

$$x_- = \liminf_{k\to\infty} \sum_{n=1}^{k} b_n \qquad x_+ = \limsup_{k\to\infty} \sum_{n=1}^{k} b_n.$$

**- Double sequence**

A function $a$ on $Z^+ \times Z^+$ is called a double series. The statement that the *double limit* exists,

$$\lim_{m,n\to\infty} a(m,n) = a$$

means for every $\epsilon$ there exists $N$ such that $|a(m,n) - a| < \epsilon$ whenever both $m > N$ and $n > N$.

If $\lim_{m,n\to\infty} a(m,n) = a$ and if for each fixed $m$, $\lim_{n\to\infty} a(m,n)$ exists, then $\lim_{m\to\infty} (\lim_{n\to\infty} a(m,n))$ the *iterated limit* exists and has limit $a$.

Conversely, define the function $a_n$ on $Z^+$ by $a_n(m) = a(m,n), m \in Z^+$. Assume that $a_n \to a$ uniformly. If the iterated limit

$$\lim_{m\to\infty} \left( \lim_{n\to\infty} a(m,n) \right)$$

exists, then so does the double limit and has the same value.

**- Infinite products**

Let $\{u_n : n \geq 1\}$ be a sequence of complex numbers. The infinite product $\prod_{n=1}^{\infty} u_n$ is said to converge if

- there exists $N$ so that $u_n \neq 0$ for $n > N$, and
- the sequence $p_k = \prod_{n=N+1}^{k} u_n$ has a limit $p \neq 0$.

In this case of convergence

$$\prod_{n=1}^{\infty} u_n = u_1 u_2 \cdots u_N p.$$

With this definition of convergence we have, for $a_n > 0$, that the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

We say that the product $\prod_{n=1}^{\infty}(1 + a_n)$ *converges absolutely* if $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges. With this definition, absolute convergence of $\prod_{n=1}^{\infty}(1 + a_n)$ implies convergence.

- **Sequences of functions**

A sequence of functions is said to *converge pointwise* to a limit function $f$ on a set $S$ provided that for every $x \in S$, and each $\epsilon > 0$, there exists $N$, depending on both $x$ and $\epsilon$ such that

$$n > N \qquad \text{implies} \qquad |f_n(x) - f(x)|$$

If the choice of $N$ does not depend on $x$, the sequence of functions is said to *converge uniformly.*

**- Unform convergence and continuity**

If $f_n \to f$ uniformly on $S$ and each $f_n$ is continouous at a point $c$, then $f$ is continuous at $c$.
Given a sequence of functions $\{f_n : n \geq 1\}$ defined on a set $S$. For each $x \in S$, set

$$s_n(x) = \sum_{k=1}^{n} f_n(x), \qquad n = 1, 2, \ldots.$$

If $s_n \to s$ uniformly on $S$, then we say that the series $\sum_{k=1}^{\infty} f_n(x)$ converges uniformly on $S$.

**Theorem. (Weierstrass $M$-test)** Let $\{M_n : n \geq 1\}$ be a sequence of nonnegative numbers such that

$$0 \leq |f_n(x)| \leq M_n \qquad \text{for } n = 1, 2, \ldots, \text{ and every } x \in S.$$

Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $S$ if $\sum_{n=1}^{\infty} M_n < \infty$.

**- The $L^\infty$ norm**

Consider the vector space $C(S)$, the real valued continuous functions on $S$, and define the *infinity norm,*

$$||f||_\infty = \sup_{x \in S} |f(s)|.$$

Then $|| \cdot ||_\infty$ is a *norm*, meaning that

- $||f||_\infty \geq 0$ and $||f||_\infty = 0$ if and only if $f(x) = 0$ for all $x \in S$.
- $||af||_\infty = |a|||f||_\infty$ for every $a \in R$.

This norm induces a metric $\rho(f, g) = ||f - g||_\infty$. The theorems above on uniform continuity show that $(C(S), \rho)$ is a complete metric space.

**- Integration and differentiation**

Many of the theorems on uniform convergence permit the reversal of the order of taking of limits.

1. **Integration.** Let $\alpha$ have bounded variation on $[a, b]$ and assume that $\{f_n : n \geq 1\} \subset R(\alpha)$ are Riemann integrable functions. Define $g_n(x) = \int_a^b f_n(t)\ d\alpha(t),\ x \in [a, b]$. Assume there exists $f$ so that $\rho(f_n, f) \to 0$, Then
   (a) $f \in R(\alpha)$, and
   (b) $\rho(g_n, g) \to 0$ where $g(x) = \int_a^b f(t)\ d\alpha(t)$.
2. **Differentiation.** Assume that $\{f_n : n \geq 1\}$ is differentiable on $(a, b)$ and that there exist a function $g$ so that $\rho(f_n', g) \to 0$ and a point $c \in (a, b)$ so that $\{f_n(c) : n \geq 1\}$ converges. Then

(a) there exists $f$ so that $\rho(f_n, f) \to 0$, and

(b) $f$ is differentiable with derivative $g$.

### - Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a *power series* centered at $z_0$. Define

$$\lambda = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \qquad r = \frac{1}{\lambda}.$$

(Take $1/0 = \infty$ and $1/\infty = 0$.) Then by the root test, the series converges absolutely if $|z - z_0| < r$ and diverges if $|z - z_0| > r$. Furthermore:

1. The series converges uniformly on every compact subset of $B(z_0, r)$.
2. The function $f$ can be differentiated term by term for any $z \in B(z_0, r)$,

$$f'(z) = \sum_{n=1} n a_n (z - z_0)^{n-1}.$$

3. The power series for $f'$ has radius of convergence $r$.

Evaluating this expression at $z_0$ shows that $a_1 = f'(z_0)$. Repeated differentiation and evaluation yields

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

### - The Stone-Weierstrass Theorem

The Taylor polynomials gives a good approximation to $f$ when it has sufficiently many derivatives. The Stone-Weierstrass theorem gives other examples.

A collection of complex valued functions $\mathcal{A}$ on a set $E$

- is an *algebra* if it is closed under addition, multiplication, and scalar multiplication,
- *separates points* if to each pair of distinct points $x_1, x_2 \in E$ there exists $f \in \mathcal{A}$ so that $f(x_1) \neq f(x_2)$,
- *vanishes at no point of* $E$ if for each $x \in E$ there exists $g \in \mathcal{A}$ so that $g(x) \neq 0$.

For example the set of polynomials and the set of trigonometric polynomials are algebras that separate points and vanishes at no point of $R$.

**Theorem.** Let $\mathcal{A}$ be an algebra of real continuous function of a compact set $K$. If $\mathcal{A}$ separate points and vanishes at no point of $K$, then $C(K)$ is the uniform closure of $\mathcal{A}$.

Uniform closure means closure under the metric $\rho$.

## • First order ordinary differential equations

Let $V \subset R^n$ and $I = [t_0, t_f]$ and $\phi : I \times V \to R^n$. A *solution* to the initial value problem

$$y' = \phi(t, y), \qquad y(t_0) = y_0$$

is a differentiable function $f$ on $I$ such that $f(t_0) = y_0$, $f(t) \in V$ and $f'(t) = \phi(t, f(t))$. The ODE is called *autonomous* if $\phi$ is independent of $t$.

The basic estimate used to study the dependence of solutions on initial conditions is *Gronwall's inequality.*

**Theorem.** Let $f, g : [a, b) \to R$ be continuous and nonnegative. Suppose

$$f(t) \leq K + \int_a^t f(s)g(s)\ ds, \qquad K \geq 0.$$

Then

$$f(t) \leq K \exp\left(\int_a^t g(s)\ ds\right) \qquad \text{for } t \in [a, b).$$

### - Linear ODEs

A *linear system* is one in which $\phi(t, y) = A(t)y + g(t)$. It is called *homogeneous* if $g(t) = 0$ for all $t$. For a homogeneous autonomous linear system $y' = Ay$, a solution is

$$y(t) = \exp(t - t_0)Ay_0.$$

By Gronwall's inequality, this solution is unique.

For nonautomonous systems having continuous $A$, the solutions of $y' = A(t)y$ form a vector space of dimension $n$ over the complex numbers. An $n \times n$ matrix whose columns are linearly independent solutions is called a *fundamental matrix.* Once this matrix valued function has been found, we can find a solution to the non-homogeneous system by *variation of constants.*

**Theorem.** If $\Phi$ is a fundamental matrix of $y' = A(t)y$ on $I$, then the function

$$\psi(t) = \Phi(t) \int_{t_0}^t \Phi^{-1}(s)g(s)\ ds$$

is the unique solution of the nonhomogenous linear system above with initial condition $\psi(t_0) = 0$.

### - Existence of solutions and iteration techniques

**Lemma.** Let $\phi$ be Lipschitz in $y$ uniformly in $t$ and assume that $B(y_0, r) \subset V$. Choose $M$ so that $|\phi(t, y)| \leq M$ for $(t, y) \in I \times B(y_0, r)$. Set $t_0 \in I$ and $\delta = r/M$. Then there is a unique $C^1$ function $y(t), t \in (t_0 - \delta, t_0 + \delta)$ satisfying $y(t) \in B(y_0, r)$ that is a solution to the ordinary differential equation.

The differential equation is equivalent to $y(t) = y_0 + \int_{t_0}^t \phi(t, y(s))\ ds$. The *Picard iteration* technique begins by seting $y_0(t) = y_0$ and defining inductively

$$y_{n+1}(t) = y_0 + \int_{t_0}^t \phi(t, y_n(s))\ ds.$$

Let $K$ be the Lipschitz constant. By induction, we find that

$$|y_{n+1}(t) - y_n(t)| \leq MK^n|t - t_0|^{n+1}/(n+1)!$$

Thus, $\{y_n; n \geq 0\}$ is a Cauchy sequence in the $L^\infty$ norm. Consequently, the limit $y$ is a continuous curve that is a solution. To check uniqueness, let $\tilde{y}$ be another solution. Then check that $|y_n(t) - \tilde{y}(t)| \leq MK^n|t - t_0|^{n+1}/(n+1)!$ to see that $\tilde{y} = y$.

**- Flow Boxes**

We now consider only autonomous systems.

**Definition.** A *flow box* of $\Phi$ at $y$ is a triple $(U_0, a, \Phi)$ in which:

1. $U_0$ is open, $y \in U_0$, and $a \in (0, \infty]$.
2. $\Phi \in C^\infty(I_a \times U_0, R^n)$ where $I_a = (-a, a)$.
3. For each $u \in U_0$, $y_u(t) = \Phi(t, u)$ is a solution to the ODE.
4. For each $t \in (-a, a)$, $\Phi_t(u) = \Phi(t, u)$ is a diffeomorphism onto its range.

If $\phi$ is $C^\infty$, then for every $y$, there is a flow box of $\Phi$ at $y$. Flow boxes are unique in the sense that if $(U_0, a, \Phi)$ and $(\tilde{U_0}, \tilde{a}, \tilde{\Phi})$ are two flow boxes at $y$, then $\Phi$ and $\tilde{\Phi}$ are equal on $(U_0 \cap \tilde{U_0}) \times (I_a \cap I_{\tilde{a}})$.

If $t_1, t_2 \in (-a, a)$ are chosen so that $-a < t_1 + t_2 < a$, then

1. $\Phi_0$ is the identity map.
2. $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$.
3. Set $U_t = \Phi(U_0)$. If $U_t \cap U_0 \neq \emptyset$, then $\Phi_t|_{U_{-t} \cap U_0} : U_{-t} \cap U_0 \to U_t \cap U_0$ is a diffeomorphism with inverse $\Phi_{-t}|_{U_t \cap U_0}$.

**- Stability around fixed points**

A point $y$ is called *positively (negatively) complete* if the solution can be extented to $+\infty$ $(-\infty)$ and *complete* if it is both positively and and negatively complete.

**Definition.** Let $\phi$ be continuously differentiable.

1. A point $y_0$ is called a *critical point*, (also called a *singular* or *equilibium point*) if $\phi(y_0) = 0$.
2. The eigenvalues of the Jacobian matrix $D\phi(y_0)$ are called the *characteristic exponents* of $\phi$ at $y_0$.
3. A critical point $y_0$ is called *Liapunov stable* if for every neighborhood $U$ of $y_0$, there is a neighborhood $V$ of $y_0$, such that if $y \in V$, $y$ is positively complete, and $\Phi_t(y) \in U$ for all $t > 0$.
4. A critical point $y_0$ is called *asymptotcially Liapunov stable* if there is a neighborhood $V$ of $y_0$ such that $y$ is positively complete and for every $y \in V$, $\Phi_t(V) \subset \Phi_s(V)$ if $t > s$, and

$$\lim_{t \to \infty} \Phi_t(V) = \{y_0\}.$$

Asymptotical stability implies stability. If $\Phi$ is continuously differentiable and if the characteristic exponents of $y_0$, a critical point of $\Phi$, have strictly negative real parts, then $y_0$ is asymptotically stable.