

Integration Workshop

Analysis Problems

August 6-10, 2004

CALCULUS

• Multivariable differential calculus

1. Let f and $\partial f/\partial y$ be continuous on $[0, 1] \times [0, 1]$ and assume that $p, q : [0, 1] \rightarrow [0, 1]$ are differentiable. Define

$$F(y) = \int_{p(y)}^{q(y)} f(x, y) dx, \quad y \in [0, 1].$$

Use the chain rule to find $F'(y)$. *Hint.* Consider $G(x_1, x_2, x_3) = \int_{x_1}^{x_2} f(t, x_3) dt$.

2. Define $C^1(R^2, R^3)$ to be the continuously differentiable functions from R^2 to R^3 and $C_0^1(R^2, R^3)$ to be those functions that vanish at the origin. For $f, g \in C^1(R^2, R^3)$, define $f \sim g$ if

$$\lim_{x \rightarrow 0} \frac{f(x) - g(x)}{x} = 0.$$

Show that this relation is an equivalence relation. Describe the algebraic and geometric properties of the spaces $C^1(R^2, R^3)/\sim$, $C_0^1(R^2, R^3)$, and $C_0^1(R^2, R^3)/\sim$.

3. A function $f : S \rightarrow R^n$ is *homogeneous of degree p* if

$$f(\lambda x) = \lambda^p f(x) \quad \text{for every } \lambda \in R \text{ and every } x \in S \text{ for which } \lambda x \in S.$$

If S is open and f is differentiable at x show that

$$x \cdot \nabla f(x) = pf(x).$$

Hint. Define $g(\lambda) = f(\lambda x)$ and compute $g'(1)$.

Conversely, show that if $x \cdot \nabla f(x) = pf(x)$ for all $x \in S$, then f is homogeneous of degree p .

4. Let S be an open connected set subset of R^n . Let $f : S \rightarrow R^n$ be differentiable at each point of S . If the total derivative vanishes for each $c \in S$, then f is constant on S .
5. let $GL(n)$ be the invertible matrices on R^n and define $\iota : GL(n) \rightarrow GL(n)$ by $\iota(M) = M^{-1}$. Find the total derivative of ι at A .

• **Implicit functions and extremum problems**

1. Show that the systems of equations

$$\begin{aligned}3x + y - z + u^2 &= 0 \\x - y + 2z + u &= 0 \\2x + 2y - 3z + 2u &= 0\end{aligned}$$

can be solved for x, y, u in terms of z ; for x, z, u in terms of y ; for y, z, u in terms of x ; but not for x, y, z in terms of u .

2. Find the maximum of $(x_1 \cdots x_n)^2$ under the restriction

$$x_1^2 + \cdots + x_n^2 = 1.$$

Use this to prove that the harmonic mean is less than the arithmetic mean.

$$(a_1 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + \cdots + a_n).$$

3. State conditions of f and g that will ensure that the equations

$$x = f_1(u, v), \quad y = f_2(u, v)$$

can be solved for u and v in a neighborhood of (x_0, y_0) . Call the solution

$$u = g_1(x, y), \quad v = g_2(x, y)$$

and let $J = \partial(f_1, f_2)/\partial(u, v)$. Show that

$$\frac{\partial g_1}{\partial x} = \frac{1}{J} \frac{\partial f_2}{\partial v}, \quad \frac{\partial g_1}{\partial y} = -\frac{1}{J} \frac{\partial f_1}{\partial v}, \quad \frac{\partial g_2}{\partial x} = -\frac{1}{J} \frac{\partial f_2}{\partial u}, \quad \frac{\partial g_2}{\partial y} = \frac{1}{J} \frac{\partial f_1}{\partial u},$$

4. Give a simpler formulation for the second derivative test for extrema if the domain of the function is a subset of the plane.

• **Multivariable Riemann integral**

For a random variable X , the *cumulative distribution function* $F_X(x) = P\{X \leq x\}$. X is called continuous if F_X is differentiable. The derivative f is called the *density function*. Then $P\{X \in A\} = \int_A f(x) dx$. The *joint cumulative distribution function* for (X, Y) ,

$$F_{(X,Y)}(x, y) = P\{X \leq x, Y \leq y\}$$

is called continuous if there exists a density function $f_{(X,Y)}$ so that

$$F_{(X,Y)}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{(X,Y)}(u, v) du dv.$$

X and Y are called independent if $f_{(X,Y)}$ factors into a function of x and a function of y .

1. Let g be continuous and strictly increasing on the support of f_X . Find the density of $g(X)$. Generalize this.
2. Let X and Y be independent with densities f_X and f_Y . Find the density of f_{X+Y} .
3. Let X and Y be independent standard normal random variables, i.e., having density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

Find the joint density for $(X/Y, X^2 + Y^2)$. Find the density for X/Y and for $X^2 + Y^2$. Generalize the second formula to n independent standard normals.

4. Let X and Y be independent exponential random variables, i.e., having density

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Find the joint density of $(X+Y, X/(X+Y))$. Find the density for $X+Y$ and for $X/(X+Y)$. Generalize this to n independent exponentials.

5. Let $f(x, y) = (x - y)/(x + y)^2$. Show that

$$\int_0^1 dx \int_0^1 f(x, y) dy = - \int_0^1 dy \int_0^1 f(x, y) dx = \frac{1}{2}$$

REAL ANALYSIS

• Sequences and series

1. Let $\{A_n : n \geq 1\}$ be a sequence of sets. Give reasonable definitions of the sets $\{A_n \text{ infinitely often}\}$ and $\{A_n \text{ almost always}\}$. Show that

$$x \in \{A_n \text{ almost always}\} \text{ if and only if } \liminf_{n \rightarrow \infty} \chi_{A_n}(s) = 1.$$

$$x \in \{A_n \text{ infinitely often}\} \text{ if and only if } \limsup_{n \rightarrow \infty} \chi_{A_n}(s) = 1.$$

2. Assume that $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \dots$, and suppose that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

3. Let $\{a_n : n \geq 0\}$ be a strictly positive sequence of numbers and prove that

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{a_n} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Let $a_n = n^n/n!$. Show that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = e, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{n}{(n!)^{1/n}} = e.$$

4. Let $\{a_n : n \geq 0\}$ be a real-valued sequence of numbers and define $\bar{a}_n = (a_1 + \dots + a_n)/n$. Show that

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} \bar{a}_n \leq \limsup_{n \rightarrow \infty} \bar{a}_n \leq \limsup_{n \rightarrow \infty} a_n.$$

What conclusions can you reach?

5. Define the sequence $\{f_n; n \geq 1\}$ recursively by $f_1 = f_2 = 1$ and $f_{n+2} = f_n + f_{n+1}$. Show that

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2}.$$

Hint. Show that $f_{n+2}f_n - f_{n+1}^2 = (-1)^{n+1}$ and deduce that $|f_{n+1}/f_n - f_{n+2}/f_{n+1}| < n^{-2}$ if $n > 4$.

6. Test for convergence (p, q and r are fixed real numbers.)

$$\begin{array}{ccccc} \sum_{n=1}^{\infty} n^k e^n & \sum_{n=1}^{\infty} (\log n)^p & \sum_{n=1}^{\infty} p^n n^p & \sum_{n=1}^{\infty} \frac{1}{n^p - n^q} & \sum_{n=1}^{\infty} \frac{1}{p^n - q^n}, \\ & \sum_{n=1}^{\infty} \frac{p^n}{n^q (\log n)^r}, & \sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}, & \sum_{n=1}^{\infty} \frac{1}{n^{(1+\frac{1}{n})}}. \end{array}$$

7. Let a be a positive real number. Define a sequence $\{x_n : n \geq 0\}$ by $x_0 = 0, x_{n+1} = a + x_n^2, n \geq 0$. Find a necessary and sufficient condition on a in order that a finite limit $\lim_{n \rightarrow \infty} x_n$ exists.

8. Let a and b be fixed natural numbers, $a \geq b \geq 1$ and define

$$s_n = \sum_{k=an+1}^{bn} \frac{1}{k}.$$

Prove that

$$\lim_{n \rightarrow \infty} s_n = \log \frac{b}{a}.$$

• Sequences of Functions

1. Let $f_n(x) = x^n$. Prove that the sequence $\{f_n : n \geq 0\}$ converges pointwise but not uniformly on $[0,1]$. Assume that g is continuous on $[0,1]$ with $g(1) = 0$ and define $g_n(x) = x^n g(x)$. Prove that $\{g_n : n \geq 0\}$ converges uniformly on $[0,1]$.
2. Let f be a continuous real valued function on $[0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x)$ exists (finitely). Prove that f is uniformly continuous.
3. Let f be a continuous function on $[0, 1]$. Evaluate the following limits:

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx, \quad \lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx.$$

4. Let $\{g_n : n \geq 0\}$ be continuous functions on $[0,1]$ and assume that

$$g_{n+1}(x) \leq g_n(x) \quad \text{for each } x \in [0, 1].$$

Prove that

$$g(x) = \sum_{n=0}^{\infty} (-1)^n g_n(x)$$

converges uniformly on $[0,1]$, for all x . Prove that $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

5. Let $\{f_n\}$ be a sequence of continuous functions from $[0, 1]$ to \mathbb{R} . Suppose that $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in [0, 1]$, and also that for some constant K , we have

$$\left| \int_0^1 f_n(x) dx \right| \leq K < \infty$$

for all n . Does

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0?$$

6. Assume that the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2. Find the radius of convergence of

$$\sum_{n=0}^{\infty} a_n^k x^n, \quad \sum_{n=0}^{\infty} a_n x^{kn}, \quad \sum_{n=0}^{\infty} a_n x^{n^2}.$$

7. Suppose the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

are given by the recurrence relation $a_0 = 0, a_1 = -1,$

$$3a_n + 4a_{n-1} - a_{n-2} = 0, \quad n = 2, 3, \dots$$

Find the radius of convergence of the series and the function to which it converges in its interval of convergence.

• **First order ordinary differential equations**

1. Let $f(x)$ be a real valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that $\lim_{x \rightarrow \infty} f(x)$ exists and is less than $1 + \frac{\pi}{4}$.

2. Let J_0 be an $n \times n$ matrix that has the value 1 above the diagonal and 0 elsewhere.

(a) Find $\exp(tJ_0)$.

(b) Verify that the identity matrix I and J_0 commute.

(c) Use this to find $\exp(t(\lambda I + J_0))$.

(d) Describe the unique solution to the differential equation $y' = Ay$ where A is a matrix in Jordan canonical form.

3. Solve $y' = Ay + g(t)$ with

(a)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad g(t) = \begin{pmatrix} 2e^t \\ t^2 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \quad g(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Check Picard iteration on the equation $y' = Ay$.

5. Let y_i be solutions on $[t_0, t_f]$ to $y' = \phi_i(t, y)$ such that $y_i(t_0) = y_i^0, i = 1, 2$. Suppose that $\|f - g\|_{\infty} < \epsilon$, then for some $K > 0$, show that

$$|y_1(t) - y_2(t)| \leq (|y_1^0 - y_2^0| + \epsilon(t_f - t_0)) \exp(K(t - t_0)).$$