INTEGRATION WORKSHOP 2005
TOPOLOGY EXERCISES

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1. BASICS

1.1 Find all the different topologies (up to homeomorphism) on the 2-element and 3-element sets.

1.2 Show that a map between two metric spaces $f : (X, d) \to (Y, d')$ is continuous at a point $p \in X$ if for any neighbourhood M of $f(p)$ its pre-image $f^{-1}(M)$ is a neighbourhood of p.

1.3 A topological space X is called a T_1-space if for any two different points $x, y \in X$ there exists an open set U which contains x but does not contain y. Prove that a space X is a T_1-space if and only if any subset consisting of a single point is closed. Also find an example of a topological space which is a T_1-space, but not Hausdorff.

1.4 Let X_α be a collection of topological spaces indexed by a set I. We define a topology on the Cartesian product $\prod_{\alpha \in I} X_\alpha$ as follows: a basis is given by sets of the form $\prod U_\alpha$, where $U_\alpha \subset X_\alpha$ is open and for all but finitely many α, $U_\alpha = X_\alpha$. Prove that this is the weakest topology (with fewest open sets) such that the projections $X \to X_\alpha$ are continuous.

1.5 Define the *profinite* topology on $\mathbf{Z}$ in which the open sets are the empty set and unions of arithmetic progressions. Show that an arithmetic progression is also a closed set in this topology. Show that if there were only finitely many primes, then the set $\{-1, 1\}$ would be open. Then show that this set is not open and conclude that there are infinitely many primes.

1.6 Let T^∞ be the product of countably many copies of the unit circle with the product topology. Define the map $\phi : \mathbf{Z} \to T^\infty$ as follows:

$$\phi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), ...).$$

Show that this map is injective and the induced topology on $\mathbf{Z}$ coincides with the profinite topology.

1.7 A topological space X is said to satisfy the *first axiom of countability* if for each point $x \in X$ there is a countable basis for the complete system of neighbourhoods at x. The space X is said to satisfy the *second axiom of countability* or also is called *completely separable* if it possesses a countable base for its open sets. Show that the second axiom of countability implies the first.

1.8 A subspace A of X is called *dense* if the closure of A coinsides with X. A topological space X is called *separable*, if there is a countable dense subset. Show that if X satisfies the second axiom of countability, then X is separable.

1.9 Show that the product of copies of the unit inteval indexed by the unit interval, $\prod_{\alpha \in [0,1]} [0, 1]$ is not first countable.

2. CONNECTEDNESS

2.1 Show that the Cantor set is totally disconnected.

2.2 Find all the different topologies, up to homeomorphism, on a 4-element set, which make it a connected topological space.

2.3 Show that $\mathbf{R}$ and $\mathbf{R}^2$ are not homeomorphic. *Hint*: use the notion of a connected set.

2.4 Prove that each connected component of a topological space X is closed.

2.5 Show that if A is both open and closed non-empty connected subset of a topological space X, then A is a connected component.

2.6 Show that if a topological space has finitely many connected components, then each of them is open and closed.

2.7 Show that if K is the Cantor set, then the complement of $K \times K$ in the unit square $[0,1] \times [0,1]$ is path-connected.

2.8 Let T^∞ be the product of countably many copies of the unit circle. Let S be the subset of T^∞ defined as follows:

$$S := (z_1, z_2, z_3, ...) : \quad z_{n+1}^2 = z_n.$$

Show that S is connected, but not path-connected.

2.9 Let the group $\mathbf{R}$ act on $\mathbf{R}^2$ by

$$t.(x, y) = (x, y + tx).$$

Prove that the quotient space with the quotient topology is not Hausdorff, but is the union of two disjoint Hausdorff subspaces. Also show that the quotient space is a T_1-space.

3. COMPACTNESS

3.1 Show that if X is compact and Y is Hausdorff and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

3.2 A space is *locally compact* if every point has a compact neighbourhood. There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. Namely, if X is locally compact Hausdorff, let $\bar{X} = X \cup \{\infty\}$. The open sets of $\bar{X}$ are the open sets of X together with the sets $(X \setminus K) \cup \{\infty\}$, where K is a compact subset of X. Prove that $\bar{X}$, called the *one point compactification*, is a compact Hausdorff space.

3.3 In a metric space (X, d), a sequence $a_1, a_2, ...$ of points of X is called a *Cauchy sequence* if for each $\varepsilon > 0$ there is a positive integer N such that $d(a_n, a_m) < \varepsilon$ whenever $n, m > N$. A metric space is called *complete* if every Cauchy sequence in X converges to a point of X. Show that a compact metric space is complete.

3.4 A metric space X is called *totally bounded* if for any $\varepsilon > 0$ it can be covered by finitely many ε-balls. Prove that X is compact if and only if X is complete and totally bounded.

3.5 Consider the space $\ell^2(\mathbf{R})$, which is the space of sequences $(x_1, x_2, x_3, ...)$ with convergent sums of squares. The *weak topology* in this Hilbert space is the weakest topology in which all maps $f : \ell^2(\mathbf{R}) \to \mathbf{R}$ of the form

$$f(x) = \sum_{i=1}^{\infty} a_n x_n, \quad \text{for some } (\mathrm{a}_1, \mathrm{a}_2, ...) \in \ell^2(\mathbf{R})$$

are continuous. Show that this topology is weaker than the standard metric topology.

3.6 Show that any closed convex bounded set in $\ell^2(\mathbf{R})$ (e.g. any closed ball) is compact in weak topology.

3.7 Prove that any set in $\ell^2(\mathbf{R})$, which is compact in weak topology is closed and bounded. Also show that the converse is not true.