INTEGRATION WORKSHOP 2006
LINEAR ALGEBRA EXERCISES

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1. Vector spaces and linear transformations

You may assume that all vector spaces are finite dimensional.

1.1. Let U_1 and U_2 be subspaces of a vector space V. Prove that $\dim U_1 + \dim U_2 = \dim(U_1 \cap U_2) + \dim(U_1 + U_2)$.

1.2. Suppose that $T_1 : V \to W$ and $T_2 : W \to X$ are linear transformations, and choose bases $\{v_i\}$ for V and $\{w_i\}$ for W. Let A_1 and A_2, respectively, be the matrices of T_1 and T_2 with respect to these bases. Show that the matrix for $T_2 \circ T_1 : V \to X$ with respect to the basis $\{v_i\}$ is the matrix product A_2A_1. (*Note:* this "explains" the formula for matrix multiplication, and proves that matrix multiplication is associative.)

1.3. A system of linear equations may have no solutions, a unique solution, or many solutions. Explain this in terms of the image and kernel of a linear transformation.

1.4. Let V and W be vector spaces and let $\mathrm{Hom}(V, W)$ be the set of linear transformations $V \to W$. Prove that $\mathrm{Hom}(V, W)$ has a natural vector space structure.

Choose bases $e_1, \dots, e_m$ for V and $f_1, \dots, f_n$ for W and let $\ell_1, \dots, \ell_m$ be the dual basis for V^*. For $1 \le i \le n$ and $1 \le j \le m$ prove that the formula

$$E_{ij}(v) = \ell_j(v)f_i$$

defines an element $E_{ij} \in \mathrm{Hom}(V, W)$.

Prove that the E_{ij} form a basis of $\mathrm{Hom}(V, W)$ and conclude that $\dim \mathrm{Hom}(V, W) = (\dim V)(\dim W)$. If you know about tensor products, this proof allows you to show that $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

To what extent does the above work if we drop the finite dimensional hypothesis?

1.5. An element $T \in \mathrm{Hom}(V, V) = \mathrm{End}(V)$ is called *nilpotent* if $T^n = 0$ for some n. Prove that T is nilpotent if and only if there exists a basis of V in which the matrix of T is strictly upper triangular, i.e., upper triangular with zeroes on the diagonal. Conclude that if $T^n = 0$ for some n, then $T^{\dim V} = 0$.

1.6. An element $T \in \mathrm{Hom}(V, V) = \mathrm{End}(V)$ is called *semi-simple* if there exist one-dimensional subspaces $V_1, \dots, V_n$ of V such that $V \cong \oplus V_i$ and $T(V_i) \subset V_i$ for all i. (This is not the standard definition unless the ground field is algebraically closed, but we'll ignore this for now.) Prove that V is semi-simple if and only if there exists a basis of V in which the matrix of T is diagonal.

1.7. Let T be any element of $\mathrm{Hom}(V, V) = \mathrm{End}(V)$. Prove that there exists a basis of V in which the matrix of T is upper triangular if and only if there exist subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$ and $T(V_i) \subset V_i$.

Recall or prove that if the ground field is $\mathbb{C}$ then T has an eigenvector. (Hint: Let v be any non-zero vector and consider the $n+1$ vectors $v, TV, T^2v, \ldots, T^nv$. They must be linearly dependent, so ...)

Prove by induction on the dimension of V that the equivalent conditions in the first paragraph are always satisfied if the ground field is $\mathbb{C}$. Give an example of a linear transformation T over some other field for which the conditions are not satisfied.

1.8. Prove that for an infinite-dimensional vector space V, the canonical injection $V \to V^{**}$ is not an isomorphism.

1.9. If $T : V \to W$ is a linear transformation and $T^* : W^* \to V^*$ is its transpose, prove that the kernel of T is the orthogonal complement of the image of T^*, in other words,

$$\ker T = \{v \in V \mid \ell(v) = 0 \text{ for all } \ell \in \mathrm{Im}\, T^*\}.$$

Similarly, the image of T is the orthogonal complement of the kernel of T^*.

1.10. Prove that if $W \subset V$ is a subspace with $\dim W = \dim V - 1$ ("codimension 1") then there exists an element $\ell \in V^*$ such that W is the kernel of ℓ. Generalize to an arbitrary subspace $W \subset V$.

1.11. Suppose that the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, \ldots, m$$

has real coefficients: $a_{ij}, b_i \in \mathbb{R}$. Prove that if the system has a solution with $(x_j) \in \mathbb{C}^n$, then it has a solution with $(x_j) \in \mathbb{R}^n$. In this case, is every solution in $\mathbb{R}^n$?

1.12. Let V be the vector space of polynomials of degree $\leq d$. Assuming $d \geq 3$, prove in two different ways that the set W of polynomials divisible by $(x-2)^2(x-3)$ has dimension $d - 2$.

1.13. Convince yourself that Gaussian elimination leads to an algorithm for writing an arbitrary matrix as a product LPU where L is lower triangular, P is a permutation matrix, and U is upper triangular.

2. Endomorphisms and the Jordan form

In this section we assume that all vector spaces are finite dimensional over $\mathbb{C}$.

2.1. Give a recipe for the sizes of the Jordan blocks of an endomorphism $T : V \to V$ in terms of the dimensions of $\ker(T - \lambda)^i$ where λ is an eigenvalue of T and $i = 1, 2, \ldots$.

2.2. Give necessary and sufficient conditions for an endomorphism to have a square root. I.e., given $T : V \to V$, when is there an endomorphism U such that $U^2 = T$?

2.3. Prove that every $n \times n$ complex matrix is similar to its transpose.

2.4. Prove that the set of diagonalizable $n \times n$ matrices is dense in $\mathbb{C}^{n^2}$, in the following sense: Given A, for every $\epsilon > 0$ there is a matrix B all of whose entries are of absolute value $< \epsilon$ such that $A + B$ is diagonalizable.

2.5. Prove that if $T^m = I$ for some $m \geq 1$ then T is diagonalizable. More generally, show that if $f(T) = 0$, where f is a polynomial with distinct roots (and the constant term a_0 of f is interpreted as $a_0 \cdot I$), then T is diagonalizable. Is the converse true?

2.6. (Uniqueness of the abstract Jordan decomposition) Suppose that $T = S+N = S' + N'$ are two abstract Jordan decompositions of the linear transformation T. Prove that $SS' = S'S$ and $NN' = N'N$.

Prove that $S-S'$ is semi-simple and $N-N'$ is nilpotent. (These statements would be false in general if S and S' did not commute or N and N' did not commute.) Conclude that $S = S'$ and $N = N'$.

2.7. Let $T : V \to V$ be an endomorphism. If $v \in V$, the *cyclic subspace* generated by v is the span of v, Tv, T^2v, ... Given the Jordan form of T, determine the possible dimensions of cyclic subspaces (for various $v \in V$), and the possible Jordan forms for T restricted to a cyclic subspace.

2.8. The Vandermonde determinant of order n is the determinant

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Show that

$$V_n = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

(*Hint:* show that $(x_j - x_i) \mid V_n$ for all $i < j$, and then compare degrees.)

2.9. Use the Jordan form to determine a set of representatives of the conjugacy classes of $GL_2(\mathbb{C})$ and $GL_2(\mathbb{R})$. If you're feeling adventurous, try the same thing for $GL_2(\mathbb{Z}/p\mathbb{Z})$, where p is an odd prime.

2.10. Determine the set of matrices commuting with a given matrix. You may want to consider several special cases first: diagonal matrices with distinct eigenvalues, diagonal matrices with a repeated eigenvalue, a single Jordan block, ...

3. Bilinear forms

3.1. Use the Gram-Schmidt process to prove that every invertible complex matrix can be written as the product of a unitary matrix and an upper triangular matrix. (“QR decomposition”) What is the real version?

3.2. Prove that an invertible real matrix A can be written as QP where P is symmetric positive definite and Q is orthogonal. (“polar decomposition”) Hint: Consider A^tA, which is a symmetric positive definite matrix, and take its square root. What is the complex version?

3.3. Suppose $(\cdot,\cdot) : V \times W \to \mathbb{C}$ is a bilinear form. Define the left kernel V_0 by

$$V_0 = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$$

and similarly for the left kernel W_0. Prove that $(\cdot,\cdot)$ induces a non-degenerate bilinear form on $(V/V_0) \times (W/W_0)$ and in particular, $\dim(V/V_0) = \dim(W/W_0)$.

3.4. A *quadratic form* on a vector space V over a field F (with characteristic $\neq 2$) is a function $q : V \to F$ such that $q(av) = a^2q(v)$ for all $v \in V$, $a \in \mathbb{C}$ and such that

$$(v, w) = \frac{1}{2}\left(q(v + w) - q(v) - q(w)\right)$$

is bilinear. If $(\cdot,\cdot)$ is a bilinear form on V, prove that $q(v) = (v, v)$ is a quadratic form on V and that every quadratic form on V arises in this way. Do q and $(\cdot,\cdot)$ uniquely determine each other?

3.5. Let $q : V \to \mathbb{R}$ be a quadratic form on the real vector space V, and $L \subset V$ a lattice (a discrete additive subgroup of full rank of V). Suppose that q satisfies the following properties:

(i) For $P \in L$, $q(P) = 0$ if and only if $P = 0$.
(ii) For every constant C, the set $\{P \in L : q(P) \leq C\}$ is finite.

Show that q is positive definite on V.

3.6. Let P_d be the space of polynomials of degree $\leq d$. Compute the signature of the symmetric bilinear form $(f, g) = (fg)^{(d)}(0)$. (The right hand side is the d-th derivative of the product fg evaluated at zero.)

3.7. Let P_2 be the space of real polynomials of degree ≤ 2 equipped with the Euclidean form $(f, g) = \int_0^1 fg$. Find an orthonormal basis of P_2. Is it possible to find a basis that consists entirely of monic polynomials?

3.8. For any complex matrix M, write $M = A + Bi$, where A and B are real matrices. What are the conditions on A and B to make M Hermitian? Skew-Hermitian? Unitary?

3.9. The following statement is false: if $T : V \to V$ is an endomorphism of a finite dimensional complex vector space, then there exists a positive definite Hermitan inner product on V with respect to which T is normal. Explain why and then add a hypothesis to make the conclusion true. What happens if “normal” is changed to “unitary” or “hermitian”?

3.10. Prove that the set of unitary $n \times n$ matrices is a compact subset of $\mathbb{C}^{n^2}$. What can you say about the topology of the subset of $\mathbb{C}^{n^2}$ of Hermitian matrices?