

Linear Algebra Lecture Topics

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Lecture 1

- Definitions: vector spaces and subspaces, quotients, direct sums; linear transformations, kernel, image; basis, dimension, rank; dual vector space.
- Techniques: coordinates and matrix with respect to a basis; change of basis; computing rank and null space of a matrix; solving linear systems.
- **Theorem.** If $T : V \rightarrow W$ is a linear transformation, then T induces an isomorphism $V/\ker(T) \xrightarrow{\sim} \text{im}(T)$. Therefore, for a linear transformation, its rank (the dimension of its image) plus its nullity (the dimension of its kernel) is equal to the dimension of its domain.
- **Theorem (Gaussian Elimination).** If $T : V \rightarrow W$ is a linear transformation, then there is a basis $\{v_i\}$ of V and a basis $\{w_i\}$ of W with respect to which T has the form

$$\begin{pmatrix} * & 0 & 0 & \cdots \\ 0 & * & 0 & \cdots \\ 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The number of nonzero pivots (*) is the rank of T .

- **Theorem.** If V is a vector space, the map from V to V^{**} that takes

$$v \mapsto \{f : \phi \mapsto \phi(v)\}$$

is an injective linear transformation. If V has finite dimension n , then V^* also has dimension n ; hence, in the finite-dimensional case, V and V^* are isomorphic, and V is canonically isomorphic to V^{**} .

Lecture 2

- Definitions: determinant; eigenvalues and eigenvectors; characteristic and minimal polynomials; similar matrices; semi-simple and nilpotent.
- Techniques: computing determinants; finding eigenvalues and eigenvectors; change of basis and diagonalization; Jordan normal form.
- **Theorem (Jordan Form).** Suppose $T : V \rightarrow V$ is an endomorphism of the vector space V . Then there is a basis $\{v_i\}$ of V , a partition of which decomposes V into a direct sum $\bigoplus V_i$, such that restricted to each summand V_i , T has the form of a Jordan block:

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

- **Theorem (Abstract Jordan Form).** Any endomorphism $T : V \rightarrow V$ of the vector space V can be written as a sum $T = S + N$, where S is semi-simple, N is nilpotent, and $SN = NS$.

Lecture 3

- Definitions: bilinear forms and pairings, non-degeneracy; symmetric, alternating, and Hermitian forms; orthogonal and orthonormal vectors; orthogonal, unitary, symmetric, Hermitian, and normal operators.
- Techniques: change of basis for a bilinear form; Gram-Schmidt process to find an orthonormal basis; classification of forms over \mathbb{R} and \mathbb{C} .
- **Theorem.** Every bilinear form can be represented uniquely as a sum of a symmetric bilinear form and an alternating bilinear form.
- **Theorem.** If $f : V \times V \rightarrow \mathbb{R}$ is a symmetric bilinear form, where V is a real vector space, then there is a basis of V with respect to which f is diagonal. Similarly, if $g : V \times V \rightarrow \mathbb{C}$ is a Hermitian form, where V is a complex vector space, then there is a basis of V with respect to which g is diagonal.
- **Theorem.** Suppose $T : V \rightarrow V$ is a normal linear transformation, where V is an inner product space over \mathbb{C} . The T is Hermitian if and only if all of its eigenvalues are real; and T is unitary if and only if all of its eigenvalues have absolute value 1.