Topology Lectures

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Abstract

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1 Introduction to topology

1.1 Topology of $\mathbb{R}$

We want to study continuity. We have the definition of a continuous function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, which is that for any $x_0 \in D$,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We also have the notion of open and closed sets. An open set is a set U such that for any $x \in U$ there is a ball centered at x contained in U. A closed set is a set which contains all of its limit points, i.e. if $x_i \in F$ then $\lim_{i\to\infty} x_i \in F$. We recognize the following facts about open and closed sets and continuous functions f:

- Every open subset of $\mathbb{R}$ is the complement of a closed set and every closed subset is the complement of an open set. That is because if U is open and F is closed, we have for any $x_i \in U^C$, if $\lim x_i \in U$ then a ball around $\lim x_i$ is in U and hence some x_i is in U, a contradiction. Similarly, if $x \in F^C$ and every ball centered at x contained a point x in F, then there would be a sequence x_i whose limit is x and all of whose elements are in F. Thus $x = \lim x_i$ is in F since F is closed, which is a contradiction.
- If F is closed, then $f^{-1}(F)$ is closed. This is because if $x_i \in f^{-1}(F)$, and $x_\infty = \lim_{i\to\infty} x_i$ then $\lim_{i\to\infty} f(x_i) = f(x_\infty)$. Furthermore, $f(x_i) \in F$, so $f(x_0) \in F$. Thus $x_\infty \in f^{-1}(F)$.
- If U is open, then $f^{-1}(U)$ is open. This follows since $f^{-1}\left(U^C\right)$ is closed

and thus $\left[f^{-1}\left(U^C\right)\right]^C$ is open. Now, if

$$\begin{aligned} & x \in f^{-1}\left(U^C\right)^C \\ & \quad \Leftrightarrow x \notin f^{-1}\left(U^C\right) \\ & \quad \Leftrightarrow f(x) \notin U^C \\ & \quad \Leftrightarrow f(x) \in U \end{aligned}$$

which implies that $x \in f^{-1}(U)$, or $\left[f^{-1}\left(U^C\right)\right]^C = f^{-1}(U)$ is open.

- Every open set is the union of intervals. Arbitrary unions of open sets is open. This follows because every open set is the union of balls.

1.2 Definition of a topology

We use what we learned about the topology of $\mathbb{R}$ to define the general notions of a topology and of a continuous function. We define a topological space by specifying which sets are open. In order for this to make good sense, we must put a few conditions on which sets we may choose to be open.

Definition 1 *A topological space $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfy:*

1. *$X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$,*
2. *Arbitrary unions of sets $U \in \mathcal{T}$ are in $\mathcal{T}$, i.e. for any indexing set I, if $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,*
3. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.*

Instead of explicitly writing $U \in \mathcal{T}$, we usually say that U is open. The complements of open sets are called closed.

Definition 2 *A set F is closed if $F^C = X \setminus F \in \mathcal{T}$, i.e. if F^C is open.*

Proposition 3 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 4 *The interior of a set A, denoted $\overset{\circ}{A}$, is the largest open set contained in A. The closure of a set A, denoted $\bar{A}$, is the smallest closed set containing A.*

The interior is open since it is the union of all open sets contained in A. The closure is closed since it is the intersection of all closed sets containing A. We can characterize the closure in terms of limit points.

Definition 5 *A point $x \in X$ is a limit point of a set $A \subset X$ if every open set U containing x also contains a point $y \in A \setminus \{x\}$.*

Proposition 6 *$\bar{A}$ is equal to the union of A and its limit points.*

Proof. Let F be a closed set containing A. Then $X \setminus F$ is an open set disjoint from A, so if x is a limit point of A it cannot be in $X \setminus A$, thus all limit points are contained in $\bar{A}$ (which is the intersection of all closed sets containing A. Conversely, if $x \in \bar{A} \setminus A$ then if there were an open set U containing x but disjoint from A, then $\bar{A} \cap U^C$ is closed set strictly contained in $\bar{A}$ containing A, a contradiction since $\bar{A}$ is the smallest such set. ■

1.3 Construction of topologies

Let X be a topological space and $Y \subset X$ be a subset. We can give Y the *subspace topology* by saying a set $U \subset Y$ is open if $U = V \cap Y$ for some open set $V \subset X$. It is easy to show that this gives a topology. Think about how this gives a topology on the sphere $S^n \subset \mathbb{R}^{n+1}$.

Let X be a topological space and let $\sim$ be an equivalence relation. Recall that an equivalence relation $\sim$ is a relation satisfying the following properties:

1. (reflexivity) $x \sim x$.
2. (symmetry) $x \sim y$ implies $y \sim x$
3. (transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Then $Q = X/\sim$ denotes the set of equivalence classes of the relation. There is a natural quotient map $q : X \to Q$ given by $q(x) = [x]$. The *quotient topology* is given by letting open sets be the sets $U \subset Q$ such that $q^{-1}(U) \subset X$ is open. One can specify the topology of the circle by considering $X = [0,1]$ and the equivalence relation $0 \sim 1$.

Another way to specify a topology is with a basis or subbasis.

Definition 7 *A basis B is a collection of subsets of X such that for all $x \in X$, (1) there exists $U \in B$ such that $x \in U$ and (2) if $U, U' \in B$ and $x \in U \cap U'$, then there is another set $U'' \in B$ such that $x \in U''$ and $U'' \subset U \cap U'$. A basis generates a topology which can be described either as the union of all elements of B or such that a set $V \subset X$ is open if every point $x \in V$ has has a set $U \in B$ such that $x \in B \subset V$.*

An example of a basis is the open intervals for $\mathbb{R}$. Note that according to the first definition, the basis specifies the topology. Note that the basis itself may not be a topology unions of basis elements may not be in the basis.

Let X and Y be topological spaces. We can give $X \times Y$ a topology by taking as a basis the sets $U \times V$ where $U \subset X$ and $V \subset Y$ are open sets. Note that not all open sets can be written as $U \times V$ for some $U \subset X$ and $V \subset Y$. This construction allows us to give a topology on $\mathbb{R}^n$ from the topology on $\mathbb{R}$.

A function $d : X \times X \to \mathbb{R}$ is a metric if for all $x, y, z \in X$, it satisfies:

1. (positive definite) $d(x,y) \geq 0$ with $d(x,y) = 0$ if and only if $x = y$

2. (symmetric) $d(x,y) = d(y,x)$
3. (triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$

Recall that the ball $B(x,r) = \{y \in X : d(x,y) < r\}$. The topology generated by the basis consisting of all balls $B(x,r)$ for all $r \in (0,\infty)$ and $x \in X$ is called the *metric topology.* (Show this is a basis.)

We can also specify a topology through a subbasis.

Definition 8 *A subbasis B' for a topology on X is a collection of sets whose union is X. The topology generated by the subbasis is such that every open set is a union of finite intersections of elements of B'.*

For a subbasis, the collection of finite intersections is a basis.

1.4 Continuous maps

We know that we can consider a continuous map from $\mathbb{R}$ to $\mathbb{R}$ to be one which whose graph does not require one to lift a pencil. It is easy to see that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map, then $f^{-1}(U)$ is open if U is open. Continuous maps between topological spaces are defined to be maps such that the inverse image of an open set is open.

Note that this implies that the inverse image of a closed set is closed (why?)

Continuous maps allow us to give an equivalence of topological spaces. We say that two topological spaces are homeomorphic if there exists a continuous map between them with a continuous inverse. Such a map is called a homeomorphism. Show that the circle given by the subspace topology from $\mathbb{R}^2$ is homeomorphic to the quotient topology $[0,1]/\sim$ as above.

1.5 Separation and countability

Here we simply list some of the separation and countability properties.

Separation:

- Hausdorff. A space is Hausdorff if for every two points $x, y \in X$, there are disjoint open sets U and V such that $x \in U$, $y \in V$, $x \notin V$, $y \notin U$. Note that a subspace of a Hausdorff space is Hausdorff but the quotient of a Hausdorff space may not be Hausdorff (example: line with two origins).
- Regular. A space is regular if one point sets are closed and for each pair of a point x and a closed set B disjoint from x there are disjoint open sets containing x and B.
- Normal. A space is normal if one point sets are closed and for each pair of disjoint closed sets A, B there are disjoint open sets containing A and B.

Hausdorff is the most important. One reason is the following.

Proposition 9 *Finite point sets in Hausdorff spaces are closed.*

Proof. It is sufficient to show that one point sets are closed since finite unions of closed sets are closed. We need only show that there are no limit points of a one point set x. Recall that a limit point of x is a point y such that every open set containing y must also contain x. But since the space is Hausdorff, there is always an open set containing y not containing x, so if $y \neq x$ then y is not a limit point. Hence $\{x\}$ is equal to its closure. ■

We note some examples of Hausdorff, regular, and normal spaces.

Countability. A set is countable if there is a bijection between it and the natural numbers. It is easy to see that the integers, the even integers, and the rational numbers are all countable sets. It is also possible to see that the real numbers between 0 and 1 form an uncountable set using Cantor's diagonal argument. Topological spaces have the following countability axioms:

- First countable. A space is first countable if every point has a countable basis, i.e. given $x \in X$ there is a countable collection of open sets $U_1, U_2, U_3, \ldots$ such that for any neighborhood V of x, there is $k \in \mathbb{N}$ such that $U_k \in V$.

- Second countable. A space is second countable if it has a countable basis for the topology. (Long line is an example which is not second countable.)

1.6 More exotic examples

- Discrete topology. All points are open sets.

- Indiscrete topology. The only open sets are X and $\varnothing$.

- Line with two origins. We consider two real lines, $\mathbb{R} = \{x : x \in \mathbb{R}\}$ and $\mathbb{R}' = \{x' : x \in \mathbb{R}\}$. The line with two origins is the quotient $\mathbb{R} \cup \mathbb{R}' / \sim$ where $x \sim x'$ if $x \neq 0$. Show that this space is not Hausdorff.

- Order topology. A total ordering on a set X is a relation $\leq$ such that for any $x, x' \in X$ we have either that $x \leq x'$ or $x' \leq x$ and both are true if and only if $x = x'$, and the relation is transitive. The order topology is the topology generated by the basis of intervals $(a, b) = \{x \in X : a \leq x \text{ and } x \leq b\}$. Products of ordered sets can be given the dictionary order. What do you think the definition of the dictionary order is? Is the order topology on $\mathbb{R}^2$ Hausdorff?

- Finite complement topology. The finite complement topology is the topology such that the complement of finite sets are open. We can also consider the countable complement topology.

- Zariski topology. Consider the following topology on $\mathbb{R}^n$. We take as the closed sets the sets

$$F(S) = \{x \in \mathbb{R}^n : f(x) = 0 \forall f \in S\}$$

where S is a set of polynomials in n variables. Show that this is a topology on $\mathbb{R}^n$. Show that any two open sets must intersect, and hence the topology cannot be Hausdorff.

2 Compactness

2.1 Closed and bounded sets in $\mathbb{R}^n$

We notice that closed and bounded subsets X of $\mathbb{R}^n$ have the following very nice properties:

- Every sequence contained in X has a limit point in X. That is, every sequence has a subsequence which converges to a point in X.
- The set X can be covered by finitely many open sets. In fact, we can do better. Given any open cover, there is a finite subset of that cover which also covers (finite subcover).
- Every continuous function on X attains a maximum and a minimum on X.

We shall use the middle definition to define compactness on a general topology.

2.2 General definitions of compactness

Definition 10 *A topological space X is* compact *if any open cover of X has a finite subcover, i.e. for any collection of open sets $\{U_i\}_{i\in I}$ with $X \subset \bigcup_{i\in I} U_i$, there exists a finite subcollection $\{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\} \subset \{U_i\}_{i\in I}$ such that $X \subset \bigcup_{j=1}^{k} U_{i_j}$.*

This may seem a rather abstract definition but it has many important properties which follow immediately. Let us take for granted for the moment that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. As a warm-up, we have the following.

Proposition 11 *A compact metric space can be covered by finitely many balls of any given radius.*

Proof. For any compact metric space X and $r > 0$, $\bigcup_{x\in X} B(x,r)$ is and open cover of X. Thus it has a finite subcover and so there exist $\{x_i\}_{i=1}^{k}$ such that $X = B(x_1, r) \cup \cdots \cup B(x_k, r)$. ■

2.3 Properties of compact spaces

In this section we see some properties of a compact set.

Proposition 12 *If X is compact and $F \subset X$ is closed, then F is compact.*
Proof. *Given an open cover $\{U_i\}_{i\in I}$ of F, there is a cover $\left\{\tilde{U}_i\right\}_{i\in F}$ such that $\tilde{U}_i$ are open in X and $U_i = \tilde{U}_i \cap F$. Consider $\left\{\tilde{U}_i\right\}_{i\in F} \cup (X \setminus F)$. This is an open cover and so it must contain a finite subcover. Since F is not contained in $X \setminus F$, it must be contained in the finite subcover $\left\{\tilde{U}_{i_j}\right\}_{j=1}^{k}$ and hence $\left\{U_{i_j}\right\}_{j=1}^{k}$.*
■

Proposition 13 *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact (with the subspace topology).*
Proof. *If $\{U_i\}_{i\in I}$ is an open cover of $f(X)$, then $U_i = \tilde{U}_i \cap f(X)$ where $\tilde{U}_i$ are open in Y. We see that $\left\{f^{-1}\left(\tilde{U}_i\right)\right\}_{i\in I}$ is an open cover of X, thus it must have a finite subcover $\left\{f^{-1}\left(\tilde{U}_{i_j}\right)\right\}_{j=1}^{k}$. But then $\left\{\tilde{U}_{i_j}\right\}_{j=1}^{k}$ must cover $f(X)$ (given $y = f(x)$, $x \in f^{-1}\left(\tilde{U}_{i_j}\right)$ and thus $y = f(x) \in \tilde{U}_{i_j}$) and thus $\left\{U_{i_j}\right\}_{j=1}^{k}$ is a finite subcover of $f(X)$.* ■

Proposition 14 *If $f : X \to \mathbb{R}$ is continuous and X is compact, then there exist x_m and x_M in X such that*

$$f(x_m) = \inf_{x\in X} f(x)$$
$$f(x_M) = \sup_{x\in X} f(x).$$

Proof. *Since $f(X)$ is compact, it must be closed and bounded and thus it contains its lower and upper bounds.* ■

Proposition 15 *If X and Y are compact, then $X \times Y$ is compact.*

Proof. We know that $\{x\} \times Y$ is homeomorphic to Y and is thus compact. Given an open cover C of $X \times Y$, we know that it must also cover $\{x\} \times Y$. Since $\{x\} \times Y$ is compact, there is a finite subcover $\{C_1, \ldots, C_k\}$ which cover $\{x\} \times Y$. We will need to show the following: Given any open set N containing $\{x\} \times Y$, there is an open set $W \subset X$ such that $\{x\} \times Y \subset W \times Y \subset N$. Suppose we have this fact, then we take $N = C_1 \cup \cdots \cup C_k$, and so there exists W as stated. Hence $W \times Y$ is covered by $\{C_1, \ldots, C_k\}$. For each x there is a W_x such that $\{x\} \times Y \subset W_x \times Y$ and $W_x \times Y$ is covered by finitely many sets. the sets $\{W_x\}_{x\in X}$ form a cover of X, and hence there is a finite subcover $\{W_1, \ldots, W_\ell\}$. The sets $\{W_1 \times Y, \ldots, W_\ell \times Y\}$ cover $X \times Y$ and each can be covered by finitely many sets in C, so the proposition is proved provided we prove the missing statement.

Suppose N is an open set containing $\{x\} \times Y$. By the definition of the product topology on $X \times Y$, around every point in $\{x\} \times Y$ there is a basis element $U \times V$ which is contained in N. Since $\{x\} \times Y$ is compact, we can take finitely many basis elements so that $\{U_1 \times V_1, \ldots, U_j \times V_j\}$ cover $\{x\} \times Y$. We may assume that each $U_i \times V_i$ intersects $\{x\} \times Y$ since otherwise it may be discarded. Now let $W = U_1 \cap \cdots \cap U_j$. Clearly this set is open and contains x_0. Finally we must show that $\{U_1 \times V_1, \ldots, U_j \times V_j\}$ actually covers $W \times Y$. Given $(w, y) \in W \times Y$, we have that $(x_0, y) \in U_i \times V_i$ for some i, but then also $w \in U_i$ (since w is in every U_i) and so $(w, y) \in U_i \times Y_i$ and the proof is done. ■

Proposition 16 *Compact subsets of a Hausdorff space are closed.*

Proof. Let K be a compact subset of a Hausdorff space X. We wish to show that X contains its limit points. Let $x \in \bar{K} \setminus K$. Since X is Hausdorff, for every $y \in K$ there are disjoint open sets U_y, V_y containing y and x respectively. The sets U_y cover K and since K is compact, there is a finite subcover $\{U_1, \ldots, U_n\}$. This implies that $\bigcap_{j=1}^{n} V_y$ is an open set (since it is a *finite* intersection) containing x disjoint from K, contradicting the fact that x is a limit point. Thus $K = \bar{K}$. ■

2.4 Heine-Borel theorem

Notice that $(0, 1]$ is not compact, because the cover $\bigcup_{k \in \mathbb{N}} \{(1/k, 1]\}$ has no finite subcovers.

Theorem 17 *(Heine-Borel)* $[0, 1]$ *is compact.*

Proof. Let $\mathcal{U}$ be a cover of $[0, 1]$. We let

$$S = \{x \in [0, 1] : [0, x] \text{ has a finite subcover in } \mathcal{U}\}.$$

Now we show that $y = \sup S$ must be 1. Observe since $\mathcal{U}$ is a cover, y is contained in some open set U, and hence the interval $(y - \varepsilon, y + \varepsilon) \subset U$ for some small $\varepsilon > 0$. This implies both that $y \in S$ since there must be some $y' \in S$, $y' > y - \varepsilon$ since y is the sup, so take the finite cover of $[0, y']$ and add in U. But this also implies that $y + \varepsilon/2 \in S$ if $y + \varepsilon/2 \in [0, 1]$, so $y = 1$. ■

Definition 18 *A subset X of $\mathbb{R}^n$ is said to be* bounded *if there exists $r > 0$ such that $X \subset B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$.*

Corollary 19 *Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.*

Proof. Closed and bounded sets are closed subsets of some compact set $[-k, k]^n$, and thus compact. Conversely, if a subset of $\mathbb{R}^n$ is compact, it must be closed since $\mathbb{R}^n$ is Hausdorff and it must be bounded because we can take the cover of $(-k, k)^n$ for all $k = 1, 2, \ldots$ and it must have a finite subcover. ■

2.5 Examples of compact spaces

- Any finite topological space is compact.
- The finite-dimensional sphere $\left\{x \in \mathbb{R}^n : |x|^2 = 1\right\}$.
- The Cantor set is compact.
- The finite-complement topology is compact.

3 Connectedness

3.1 Connected and disconnected sets in $\mathbb{R}^n$

The key property of a connected set is the intermediate value theorem, which states that if $f : [a, b] \to \mathbb{R}$ is a continuous function and $f(a) \leq r \leq f(b)$ then there exists $c \in [a, b]$ such that $f(c) = r$. Notice this is not true for functions on disconnected sets such as $(0, 1) \cup (1, 2)$.

3.2 Definition of connected

Definition 20 *A separation of a space X is a pair U, V of disjoint open subsets of X such that $X = U \cup V$. Note that the two sets U and V are both open and closed since $U = X \setminus V$ and $V = X \setminus U$. The trivial separation consists of X and $\varnothing$.*

Definition 21 *A space X is* connected *if there exist no nontrivial separations of X. Equivalently, X is connected if the only open and closed subsets of X are X and $\varnothing$ (since if $A \subset X$ is open and closed, then $X = A \cup A^C$ is a separation if neither is empty). A space which is not connected is said to be* disconnected.

Example 22 $(0, 1)$ *is connected.*

Example 23 $(0, 2) \backslash \{1\}$ *is disconnected since* $(0, 1)$ *and* $(1, 2)$ *form a nontrivial separation.*

A note on the proof of Heine-Borel: we essentially used that $[0, 1]$ is connected and showed that the set of points $y \in [0, 1]$ such that $[0, y]$ can be covered by a finite subcover is both open and closed, and hence must be everything.

3.3 Properties of connected sets

Proposition 24 *The union of connected sets which each have a common point is connected.*

Proof. Let $X = \bigcup X_i$ such that $x \in \bigcap X_i$ and X_i are connected. Now suppose that $X = U \cup V$ where U and V are disjoint open sets. Since U and V are disjoint, $x \in U$ or V. Say $x \in U$. Then $U \cap X_i$ and $V \cap X_i$ are a disjoint open cover of X_i, and so $X_i \subset U$ (since $x \in U$). This is true for all i, so $X \subset U$. ■

Proposition 25 *Let A be a connected subset of X. If $A \subset B \subset \bar{A}$ then B is connected.*

Proof. Suppose $B = U \cup V$, where U and V are disjoint and open. Then $U \cap A$ and $U \cap B$ are disjoint and open and cover A, so $A \subset U$ or $A \subset V$, say $A \subset U$. So $\bar{A} \subset \bar{U} = U$, since $U = X \setminus V$ is closed. ■

Proposition 26 *The product of connected sets is connected.*

Proof. We see that $\{x\} \times Y$ and $X \times \{y\}$ are connected. Since both contain (x, y), $V_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$ is connected. Now we see that

$$\bigcup_{x \in X} V_{x,y} = X \times Y$$

and

$$\bigcap_{x \in X} V_{x,y} = X \times \{y\} \neq \varnothing.$$

Thus $X \times Y$ is connected. ■

Proposition 27 *If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.*

Proof. If $A \subset f(X)$ is nonempty and both open and closed, then since f is continuous, $f^{-1}(A)$ is both open and closed, and hence $f^{-1}(A) = X$. But then $A = f(X)$. Thus $f(X)$ is connected. ■

Proposition 28 *(Intermediate value theorem) If X is connected, $f : X \to \mathbb{R}$ is continuous, and $f(a) \leq r \leq f(b)$ then there exists $c \in X$ such that $f(c) = r$.*

Proof. We know that $f(X)$ is a connected subset of $\mathbb{R}$. Now if there is no c such that $f(c) = r$, then we can cover $f(X)$ by the sets $(-\infty, r) \cap f(X)$ and $(r, \infty) \cap f(X)$, which are disjoint open sets. They are nonempty since one contains $f(a)$ and the other $f(b)$. This is a separation, contradicting that $f(X)$ is connected. ■

3.4 Path connected

Definition 29 *A path in X is a continuous map $\gamma : [a, b] \to X$.*

Definition 30 *A space X is* path connected *if any two points can be joined by a path.*

We shall see that path connected is weaker than connected, but first we show that path connected implies connected.

Proposition 31 *If X is path connected, then it is connected.*

Proof. We shall show that if X is not connected, then it is not path connected. If X is not connected, then there is a separation $\{U, V\}$. Given $x, y \in X$ if there were a path $\gamma : [a, b] \to X$ between them, then $\gamma([a, b])$ would be connected, which implies the path must lie entirely in U or V (otherwise U and V would form a separation for $\gamma([a, b])$), which says that there are no paths between points in U and points in V. Hence X is not path connected. ■

3.5 Components

Definition 32 *Given X, we can define an equivalence relation on X by setting $x \sim y$ if there is a connected subset containing both x and y. The equivalence classes are called* components *or* connected components *of X.*

Show that this is an equivalence relation.

Proposition 33 *The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one component.*

Proof. Let $\{C_i\}_{i \in I}$ be the components. Since the components are equivalence classes, they must be disjoint and must cover. If U is connected and $x_i \in U \cap C_i$ and $x_j \in U \cap C_j$ then $x_i \sim x_j$, which implies that $C_i = C_j$ by the definition of components. Now we must show that components are connected. Fix $x_0 \in C_i$. For any $x \in C_i$, there is a connected set A_x containing both x_0 and x since $x \sim x_0$. Thus $C_i = \bigcup_{x \sim x_0} A_x$, which implies that C_i is connected since it is the union of connected sets with a common intersection point x_0. ■

We can also look at path components.

Definition 34 *Define an equivalence relation on X by $x \sim y$ if there is a path from x to y. The equivalence classes are called* path components *of X.*