

What every incoming student is expected to know

Here is a list of topics from various areas of mathematics which we expect that incoming students know. Of course, few if any students will have a complete command of all of these topics, and so we are making the list available as an aid in preparing to enter the program. We strongly encourage each entering student to review (or learn) this material, using the references mentioned or other similar ones.

We note that the phrase “know” in the first sentence above does not mean “have seen.” Rather, we expect that students will know the definitions, the main examples, and that he or she can solve interesting problems using these concepts. The best way to test your level of understanding is to get a book on the subject and go directly to the exercises. If you can do most of them without breaking a sweat, then you “know” the topic.

Linear Algebra

1. Vector spaces and linear transformations. subspaces, quotient spaces, and direct sums. Kernel and image of a linear transformation.
2. Bases in a linear space. The matrix of a linear transformation with respect to two bases. Change of basis formula.
3. Gaussian elimination (“row reduction”) and applications to systems of equations, ranks.
4. The dual space and the transpose of a linear transformation.
5. Determinants, traces, characteristic and minimal polynomials.
6. Eigenvectors and eigenvalues. Generalized eigenvectors. Various normal forms, especially the Jordan form. The determinant, trace, and characteristic polynomial in terms of the eigenvalues.
7. Scalar products (both complex and real). Matrix of a scalar product and change of basis formula. Orthonormal bases and the Gram-Schmidt process.
8. Self-adjoint and symmetric operators. Unitary and orthogonal operators. Spectral theorems.

References: “Linear Algebra Done Right” by Axler is well-written and accurately titled. “Linear Algebra” by Hoffman and Kunze is a standard reference which is a bit more high-brow.

Vector Calculus and Differential Equations

1. The derivative of a function $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$. Linear approximation.
2. Line integrals. The multidimensional Riemann integral. Change of variables formula.
3. Div, grad, curl. Divergence theorem, Stokes’ theorem, Green’s theorem.
4. Existence and uniqueness of solutions to a first order ODE.
5. First and second order linear ODEs with constant coefficients.
6. Higher order linear ODEs and systems of first order ODEs.

References: Courant “Differential and Integral Calculus” is a classic. Another option is Apostol’s two volume “Calculus” (Wiley). You might supplement either with the differential equations books by Boyce and DiPrima or Edwards and Penney.

Complex Analysis

1. Holomorphic (analytic) functions.

2. Elementary functions and the mappings that they define.
3. Conformal mappings. Problems such as finding a conformal mapping from a quadrant to a disc.
4. The Taylor series of a holomorphic function.
5. Contour integrals. Cauchy's theorem. The Cauchy formula.
6. Poles. Meromorphic functions. Laurent series. Residues.
7. Evaluating contour integrals of meromorphic functions.
8. Using complex integration for evaluating some improper integrals.

References: Churchill and Brown "Complex Variables and Applications." Two more advanced references are Ahlfors "Complex Analysis," and Conway "Functions of One Complex Variable."

Algebra

1. Groups and homomorphisms of groups. Basic examples: permutation groups, alternating groups, cyclic groups, dihedral groups, linear groups.
2. Subgroups, normal subgroups, quotient groups, product groups.
3. Group actions on sets. Orbits and stabilizers.
4. Structure of finitely generated abelian groups.
5. Rings, ideals, quotient rings. Prime ideals and maximal ideals.
6. Principal ideals and principal ideal domains (PIDs). Examples of PIDs (integers, polynomial rings). Division algorithm and factorization in PIDs.
7. Modules. Structure of finitely generated modules over a PID.
8. Basic field theory. Finite fields.

References: Two we like are Artin "Algebra," and Dummitt and Foote "Abstract Algebra."

Real Analysis

1. Improper integrals. Convergence tests.
2. Sequences and series. Taylor series. Radius of convergence.
3. Series of functions. Uniform convergence.
4. Exchanging the order of summation and limit.
5. Differentiation and integration of series of functions.
6. Countable and uncountable sets. Axiom of choice and Zorn's lemma.
7. Axioms for \mathbf{R}

References: Most of this is in Courant (see Calculus above). Two other classics are Spivak and Apostol.

Point set topology

1. Open, closed, connected, and compact subsets of \mathbf{R}^n . Heine-Borel theorem.
2. Topological spaces and Hausdorff spaces.
3. Metric spaces as topological spaces.
4. Continuous functions.
5. Compact sets. Images of compact sets are compact. Continuous functions achieve their maximum on compact sets.
6. Connected sets. Images of connected sets are connected.

References: Munkres "Topology: A First Course" or the first two chapters of Singer and Thorpe "Lecture Notes on Topology and Geometry." See also your real analysis references.