Integration workshop 2006

Calculus problems

August 4 – 8, 2006

1 Multivariable Calculus

1. (a) Give an example of a function of two variables that is discontinuous at the origin, but whose partial derivatives at the origin exist.

   (b) Give an example of a function of two variables all of whose directional derivatives exist at the origin, but the function itself is not differentiable at the origin.

2. $f : S \to \mathbb{R}^m$ is differentiable and $Df = 0$ on S, an open subset of $\mathbb{R}^n$. Show that f is a constant on S.

3. Evaluate the derivatives of the following matrix functions:

   (a) $\text{inv} : GL(n) \to GL(n)$ given by $\text{inv}(M) = M^{-1}$.

   (b) The determinant function which maps $GL(n)$ to $\mathbb{R} - \{0\}$.

4. Show that, every point p on the sphere $x^2 + y^2 + z^2 = 1$ has a (3 dimensional) neighborhood U such that there is a smooth, one to one mapping of an open neighborhood V of the origin in R^3 such that the plane $z = 0$ maps to the surface of the sphere.

5. A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if the matrix of second partial derivatives (the Hessian) is positive semi-definite.

   If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and convex, show that

$$\frac{1}{k}[f(a_1) + f(a_2) + \cdots + f(a_k)] \geq f\left[\frac{a_1 + a_2 + \cdots + a_k}{k}\right]$$

6. Let $\mathbf{F}$ denote the vector field $(x^2 - z^2, 2xy, z)$ on $\mathbb{R}^3$. Compute in two different ways the surface integral

$$\iint_{\mathcal{T}} \mathbf{F} \cdot \mathbf{n} dA$$

where $\mathcal{T}$ denotes the surface of the tetrahedron bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x + y + z \leq 1$ and $\mathbf{n}$ denotes the outward normal to $\mathcal{T}$. Use the two answers to verify the divergence theorem.

7. Compute in two different ways the line integral

$$\oint_{\mathcal{C}} y dx - x dy + z^2 dz$$

where $\mathcal{C}$ is the intersection of the paraboloid $z = x^2 + 4y^2$ with the cylinder $x^2 + y^2 = 9$, traversed counter-clockwise when viewed from the point $x = 0, y = 0, z = 100$. Use the two answers to verify Stokes' theorem

8. If $f(x, y)$ and $g(x, y)$ are differentiable functions, and $\frac{\partial g}{\partial y} \neq 0$, we can recast the problem of extremizing f subject to $g = 0$ in terms of an unconstrained optimization of a function of a single variable.

   Show that this approach yields the same equations as using a Lagrange multiplier to impose the constraint.

2 Analysis

1. Given a set S in a metric space, let $\bar{S}$ denote the set obtained by adding all the limit points of S to S, *i.e.*, if $p \in \bar{S}$, then there is a sequence $x_n \in S$ such that $x_n \to p$. Show that $\bar{\bar{S}} = \bar{S}$, that is, the completion of a metric space is done in "one step".

2. Consider the alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

   (a) Show that, as presented, the series (really the sequence of partial sums) converges to $\ln(2)$.

(b) Given real numbers $a < b$, show that the series can be rearranged such that the sequence s_k of partial sums satisifes

$$\liminf_k s_k = a, \qquad \limsup_k s_k = b.$$

3. Show that $\mathbb{Q}[x]$, the set of all polynomials in one variable with rational coefficients is countable.

4. (a) Give an example of a function $f : [0,1] \to \mathbb{R}$ that is continuous at all the irrationals and discontinuous at all the rationals.

   (b) Show that the set of points of discontinuity of an arbitrary real valued function is always a F_σ, and the set of points of continuity is always a G_δ set.

   (c) [Optional.. hard..] There is no real valued function that is continuous on all the rationals but discontinuous on all the irrationals (Use Baire category theorem).

5. A non-negative sequence a_n is subadditive if $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{Z}^+$. If a_n is a subadditive sequence, show that

$$\lim_{n\to\infty} \frac{a_n}{n}$$

6. (a) Let f be a continuous function of $[0,1]$. Show that the following limits exist and evaluate them:

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx, \qquad n \lim_{n\to\infty} \int_0^1 x^n f(x)dx.$$

   (b) Let g be a differentiable function of $[0,1]$ such that $g(1) = 0$. Show that the following limit exists and evaluate:

$$\lim_{n\to\infty} n^2 \int_0^1 x^n g(x)dx.$$

7. Give a counter example to the following statement: A sequence of differentiable functions f_n converges uniformly to a differentiable function f. This implies that the sequence of derivatives f_n' converges to f' pointwise.

8. f_n is a sequence of uniformly bounded non-negative Riemann integrable functions, that is monotone non-decreasing on $[0,1]$, *i.e* $n \geq m$ and $x \in [0,1]$ implies that $f_n(x) \geq f_m(x)$, and $0 \leq f_n(x) \leq K < \infty$ for all n, x.

   (a) Show that the sequence $f_n(x)$ converges pointwise to a bounded function $f(x)$.

   (b) Show that the sequence of real numbers $s_k = \int_0^1 f_k(x)dx$ also converges $s_k \to s$ as $k \to \infty$.

   (c) Is it true that f is Riemann integrable, and $\int_0^1 f(x)dx = s$? Prove or give a counter example.

3 Applications

1. Define

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Show that f is *smooth*, that is, it has continuous derivatives of all orders.

2. Define

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

   (a) If radius of convergence of the above power series is r, then show that the radii of convergence for the series obtained by differentiating and integrating the above series termwise is also r.

   (b) Show that the termwise differentiated series converges uniformly on every disk of the form $|z - z_0| \leq \rho < r$. Use this to show that $f'(z)$ is given by termwise differentiating the above series, for any compact subset of the disk $|z - z_0| < r$.

3. If $f(z)$ is analytic on the closed disk $B(z_0, r)$, show that the derivatives of f at z_0 can be bounded by

$$f^{(n)}(z_0) \leq \frac{Cn!}{r^{n+1}}.$$

4. Using contour integration (or otherwise), evaluate
$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1}$$

5. A Möbius transformation , or a fractional linear transformation is a mapping of the form
$$w = f(z) = \frac{az+b}{cz+d}$$
where $a, b, c, d \in \mathbb{C}$. This can be extended to the *Riemann sphere* = $\mathbb{C} \cup \{\infty\}$ by
$$f(\infty) = a/c, \qquad f(-d/c) = \infty$$

(a) Show that the Mobius transformations form a group.

(b) Show that the Mobius transformations map circles to circles on the Riemann sphere (Note: A straight line on $\mathbb{C}$ is considered a circle through $\{\infty\}$ on the Riemann sphere)

6. A *gradient flow* is a dynamical system $\dot{x} = -\nabla V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a real valued C^2 function, that is referred to as the *potential* of the gradient flow.

(a) Show that $\frac{d}{dt}V \leq 0$ along orbits of the gradient flow.

(b) Show that the origin is an asymptotically stable fixed point for the two dimensional dynamical system
$$\dot{x} = -2x + y + x^2$$
$$\dot{y} = x - 2y$$

(c) identify the other fixed points for this system. Are they stable/asymptotically stable?

(d) Show that this dynamical system cannot have nontrivial periodic orbits.

7. Derive a series representation for the fundamental matrix $\Phi(t)$ of the linear system $y' = Ay$ using Picard iteration.

8. Solve $y' = Ay + g(t)$ with $y(0) = (0, 0)$,
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad g(t) = \begin{pmatrix} 2e^{-t} \\ te^{-2t} \end{pmatrix}$$