TOPOLOGY PROBLEMS

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1. BASICS

1.1 Find all the different topologies (up to homeomorphism) on the 2-element and 3-element sets.

1.2 Show that a map between two metric spaces $f : (X, d) \to (Y, d')$ is continuous at a point $p \in X$ if for any neighbourhood M of $f(p)$ its pre-image $f^{-1}(M)$ is a neighbourhood of p.

1.3 Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Show that $g \circ f : X \to Z$ is continuous as well.

1.4 A topological space X is called a T_1-space if for any two different points $x, y \in X$ there exists an open set U which contains x but does not contain y. Prove that a space X is a T_1-space if and only if any subset consisting of a single point is closed. Also find an example of a topological space which is a T_1-space, but not Hausdorff.

1.5 Let X_α be a collection of topological spaces indexed by a set I. We define a topology on the Cartesian product $\prod_{\alpha \in I} X_\alpha$ as follows: a basis is given by sets of the form $\prod U_\alpha$, where $U_\alpha \subset X_\alpha$ is open and for all but finitely many α, $U_\alpha = X_\alpha$. Prove that this is the weakest topology (with fewest open sets) such that the projections $X \to X_\alpha$ are continuous.

1.6 Define the *profinite* topology on $\mathbf{Z}$ in which the open sets are the empty set and unions of arithmetic progressions. Show that an arithmetic progression is also a closed set in this topology. Show that if there were only finitely many primes, then the set $\{-1, 1\}$ would be open. Then show that this set is not open and conclude that there are infinitely many primes.

1.7 Let T^∞ be the product of countably infinitely many copies of the unit circle with the product topology. Define the map $\phi : \mathbf{Z} \to T^\infty$ as follows:

$$\phi(n) = (\exp(2\pi i n/2), \exp(2\pi i n/3), \exp(2\pi i n/4), \exp(2\pi i n/5), \ldots).$$

Show that this map is injective and the induced topology on $\mathbf{Z}$ coincides with the profinite topology.

Date: August, 2007.

1.8 A topological space X is said to satisfy the *first axiom of countability* if for each point $x \in X$ there is a countable basis for the complete system of neighbourhoods at x (i.e. each neighbourhood contains a neighbourhood from the countable basis). Show that if X has a countable basis of its topology (i.e. is *second countable*), then it is first countable as well.

1.9 A subspace A of X is called *dense* if the closure of A coincides with X. A topological space X is called *separable*, if there is a countable dense subset. Show that if X satisfies the second axiom of countability, then X is separable.

1.10 Show that the product of copies of the unit interval indexed by the unit interval, $\prod_{\alpha \in [0,1]} [0,1]$ is not first countable.

1.11 (Zariski topology) Consider the topology on $\mathbb{R}^n$ in which the open sets are the empty set and the complements of the common zero levels sets of finitely many polynomials. Show that this is indeed a topology on $\mathbb{R}^n$, which is not Hausdorff.

2. CONNECTEDNESS

2.1 Show that the Cantor set is totally disconnected.

2.2 Find all the different topologies, up to homeomorphism, on a 4-element set, which make it a connected topological space.

2.3 Prove that the closure of a connected subspace is connected.

2.4 Show that $\mathbf{R}$ and $\mathbf{R}^2$ are not homeomorphic. *Hint*: use the notion of a connected set.

2.5 Prove that each connected component of a topological space X is closed.

2.6 Show that if A is a both open and closed, non-empty, connected subset of a topological space X, then A is a connected component.

2.7 Show that if a topological space has finitely many connected components, then each of them is open and closed.

2.8 A space X is called *locally path-connected*, if for each $x \in$ and every neighbourhood U of x, there exists a path-connected neighbourhood V of x contained in U. Show that if X is connected and locally path-connected, then it is path-connected.

2.9 Show that if K is the Cantor set, then the complement of $K \times K$ in the unit square $[0,1] \times [0,1]$ is path-connected.

2.10 Let the group $\mathbf{R}$ act on $\mathbf{R}^2$ by

$$t.(x,y) = (x, y + tx) \ .$$

Prove that the quotient space with the quotient topology is not Hausdorff, but is the union of two disjoint Hausdorff subspaces. Also show that the quotient space is a T_1-space.

3. COMPACTNESS

3.1 Prove that the unit sphere in $\mathbb{R}^n$ is compact.

3.2 Consider the topology on X in which the open sets are the empty set and the complements of finite subsets. Show that every subset of X is compact, although not every subset of X is closed, in general.

3.3 Let $\mathbb{RP}^n$ denote the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$, by the equivalence relation $\mathbf{x} \sim \mathbf{y}$ iff $\exists \lambda \neq 0$, s.t. $\mathbf{x} = \lambda \mathbf{y}$. Show that $\mathbb{RP}^n$ is compact.

3.4 Show that if X is compact and Y is Hausdorff and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

3.5 A space is *locally compact* if every point has a compact neighbourhood. There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. Namely, if X is locally compact Hausdorff, let $\bar{X} = X \cup \{\infty\}$. The open sets of $\bar{X}$ are the open sets of X together with the sets $(X \setminus K) \cup \{\infty\}$, where K is a compact subset of X. Prove that $\bar{X}$, called the *one point compactification*, is a compact Hausdorff space.

3.6 In a metric space (X, d), a sequence $a_1, a_2, ...$ of points of X is called a *Cauchy sequence* if for each $\varepsilon > 0$ there is a positive integer N such that $d(a_n, a_m) < \varepsilon$ whenever $n, m > N$. A metric space is called *complete* if every Cauchy sequence in X converges to a point of X. Show that a compact metric space is complete.

3.7 A metric space X is called *totally bounded* if for any $\varepsilon > 0$ it can be covered by finitely many ε-balls. Prove that X is compact if and only if X is complete and totally bounded.

3.8 Consider the space $\ell^2(\mathbf{R})$, which is the space of sequences $(x_1, x_2, x_3, ...)$ with convergent sums of squares. The *weak topology* in this Hilbert space is the weakest topology in which all maps $f : \ell^2(\mathbf{R}) \to \mathbf{R}$ of the form

$$f(x) = \sum_{i=1}^{\infty} a_n x_n, \quad \text{for some } (a_1, a_2, ...) \in \ell^2(\mathbf{R})$$

are continuous. Show that the unit closed ball in $\ell^2(\mathbf{R})$ is compact in weak topology.