# THREE LECTURES ON BASIC TOPOLOGY

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## 1. Basic notions.

Let $X$ be a set. To make a topological space out of $X$, one must specify a collection $\mathcal{T}$ of subsets of $X$, which are said to be *open* subsets of $X$, and which satisfy the following three properties:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$
2. any finite intersection of subsets from $\mathcal{T}$ is in $\mathcal{T}$
3. the union of an arbitrary collection of subsets from $\mathcal{T}$ is in $\mathcal{T}$

In this case we say that $\mathcal{T}$ is a *topology* on $X$, and $(X, \mathcal{T})$ is a *topological space.*

*Example 1.* For any $X$, let $\mathcal{T} = \{X, \emptyset\}$. This defines a topology on $X$ called the *coarse* topology.

*Example 2.* On the opposite end of the spectrum from the coarse topology, there is the so-called *discrete* topology, in which every subset of $X$ is declared to be open.

*Example 3.* For any set $X$, declare a subset $V \subset X$ open iff its complement $X \setminus V$ is finite, or empty, or equals to the whole of $X$. This is easily seen to define a topology on $X$.

Many of the examples of topological spaces come from the so-called *metric spaces.* Recall that a metric space is a set $X$ with a function (distance) $d : X \times X \to \mathbb{R}$ defined on pairs of its elements, which is

- symmetric, i.e. $d(x, y) = d(y, x)$
- positive, i.e. $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$
- and satisfies the triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$

*Example 4.* The usual Eucledian metric on the space $X = \mathbb{R}^n$.

Now we will show how a given metric $d$ defines the corresponding metric topology on a metric space $(X, d)$. First, for any point $x \in X$ and any positive number $\delta$ we define the $\delta$-ball $B(x, \delta)$ centered at $x$ as the collection of all points $y \in X$ such that $d(x, y) < \delta$.

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Next, we say that a set $U$ is a neighbourhood for a point $x \in U$ if there exists $\delta > 0$ such that $B(x, \delta) \subset U$. Finally, we declare a subset $V \subset X$ open if it is a neighbourhood for each of its points. It is not hard to check (do check!) that this indeed defines a topology $(X, \mathcal{T}_d)$ on $(X, d)$ called the metric topology associated with $d$.

*Example 5.* On any space $X$ define $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. This clearly defines a metric on $X$. The corresponding metric topology coincides with the discrete topology.

**Definition 1.1.** *A subset $C$ of a topological space $(X, \mathcal{T})$ is called* closed *iff its complement $X \setminus C$ is open, i.e. $X \setminus C \in \mathcal{T}$.*

Now we will discuss the cornerstone notion of *continuity*. The familiar definition of a continuous function on $\mathbb{R}^n$ naturally generalizes to maps between metric spaces:

**Definition 1.2.** *Let $(X, d)$ and $(Y, d')$ be two metric spaces. A map $f : X \to Y$ is said to be* continuous *at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, p) < \delta$, we have $d'(f(x), f(p)) < \varepsilon$. The map $f$ is said to be* globally continuous*, or just continuous, if it is continuous at every point of $X$.*

Another reformulation of this definition reads as follows (prove this!):

**Lemma 1.3.** *A map $f : (X, d) \to (Y, d')$ is continuous at $x \in X$ iff for every neighbourhood $V$ of $f(x)$, the subset $f^{-1}(V) \subset X$ is a neighbourhood of $x$.*

This in turn motivates the following very general definition.

**Definition 1.4.** *Let $X$ and $Y$ be two topological spaces and $f : X \to Y$ a map between them. The map $f$ is called* continuous*, iff for any open subset $V \subset Y$, its full pre-image $f^{-1}(V)$ is open in $X$.*

Note that the previous lemma can be taken as a general definition, if we define a neighbourhood of $x \in X$ as any subset of $X$ that contains an open set containing $x$. An open set then is obviously a neighbourhood of each of its points.

*Example 6.* Let $X$, $Y$ be two topological spaces, $y_0 \in Y$ and define $f : X \to Y$ by $f(x) = y_0$ for all $x \in X$ (a constant map). Clearly, $f$ is continuous.

*Example 7.* Let $X$ be a space endowed with the discrete topology and let $f : X \to Y$ be any map. Then $f$ is naturally continuous.

**Definition 1.5.** *Given a subset $S$ of a topological space $X$, we define its* closure*, $\bar{S}$, as the set of all points in $X$ which do not have neighbourhoods non-intersecting with $S$. The* boundary *$\partial(S)$ of $S$ is defined as the subset of $\bar{S}$ consisting of those points, whose every neighbourhood contains a point from $X \setminus S$.*

The following result is left as an exercise for the reader:

**Lemma 1.6.** *The closure of* $S$, $\bar{S}$ *is a closed subset of* $X$, *and* $\bar{S} \setminus \partial(S)$ *is open. Also,* $\partial(S) = \bar{S} \cap \overline{X \setminus S}$.

A very important notion in topology is the notion of *homeomorphism.*

**Definition 1.7.** *A* homeomorphism *between two topological spaces* $X$ *and* $Y$ *is a bijective continuous map* $f : X \to Y$ *such that its inverse* $f^{-1} : Y \to X$ *is continuous as well.*

*Example 8.* Any two open non-empty intervals $(a, b)$ and $(c, d)$ in $\mathbb{R}$ are homeomorphic.

To define a topology $\mathcal{T}$ on a space $X$ it is not necessary to list all the open sets in $\mathcal{T}$, but sufficient to specify a so-called *basis of topology* $\mathcal{B} = \{\mathcal{O}_\alpha\}_{\alpha \in I}$, consisting of only some open sets $\mathcal{O}_\alpha$ indexed by an indexing set $I$ such that any other open set $V \in \mathcal{T}$ can be represented as the union of some sets from $\mathcal{B}$.

*Example 9.* The standard topology on $\mathbb{R}$ has a countable basis consisting of open intervals $(p, q)$, $p < q$ with both endpoints being rational, together with the empty set. More generally, the standard topology on $\mathbb{R}^n$ has a countable basis consisting of open balls $B(p, \delta)$, where $p \in \mathbb{R}^n$ has all rational coordinates and $\delta \in \mathbb{Q}$ as well.

Next, we will study some techniques of constructing new topological spaces out of the existing ones.

Let $X$ and $Y$ be two topological spaces. We define the *product topology* on the Cartesian product $X \times Y$ as the one, which has the basis consisting of all subsets of the form $V \times U$, where $V$ is open in $X$ and $U$ is open in $Y$. This can be generalized to the product of any finite number of topological spaces.

*Example 10.* The standard topology on $\mathbb{R}^n$ is the same as the product topology on $n$ copies of $\mathbb{R}$ with its standard topology.

Next, let $S$ be a subset of a topological space $(X, \mathcal{T})$. We can endow $S$ with the so-called *subspace topology,* in which a subset $W \subset S$ is said to be open if there exists an open subset $V \in \mathcal{T}$ such that $W = V \cap S$.

*Example 11.* The induced topology on a linear subspace $\mathbb{R}^k \subset \mathbb{R}^n$ from the standard topology on $\mathbb{R}^n$ is the standard topology on $\mathbb{R}^k$.

Recall, that a relation on the set $X$ is a subset $R$ of $X \times X$. We write $x \sim y$ if $(x, y) \in R$. The relation $R$ is called an equivalence relation if the following three conditions hold:

(1) Reflexivity: $x \sim x$

(2) Symmetry: $x \sim y \Rightarrow y \sim x$
(3) Transitivity: $x \sim y,\ y \sim z\ \Rightarrow\ x \sim z$

Given a topological space $X$ with an equivalence relation $R$, one can construct the *quotient space* $Q = X/\sim$ with the *quotient topology.* Points in $Q$ represent equivalence classes of $R$ in $X$, so there is a natural surjective quotient map $q : X \to Q$. The quotient topology on $Q$ is defined as follows: a subset $U \subset Q$ is said to be open if its full pre-image $q^{-1}(U)$ is open in $X$.

*Example 12.* (Line with two origins.) Let $X = \mathbb{R} \coprod \mathbb{R}$ be the disjoint union of two copies of the real line with the standard topology. Define an equivalence relation on $X$, by saying that $x$ in the first copy of $\mathbb{R}$ is equivalent to $y$ in the second copy of $\mathbb{R}$ iff $x = y$ and $x \neq 0$.

The quotient topology in the preceding example is not, what is called, a *Hausdorff* topology.

**Definition 1.8.** *A topological space* $(X, \mathcal{T})$ *is called* Hausdorff, *if for any* $x, y \in X$, $x \neq y$, *there exist* $\mathcal{O}_x, \mathcal{O}_y \in \mathcal{T}$ *such that* $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$, *and* $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$.

In Example 12 the two origins do not have disjoint neighbourhoods. The topology from Example 3 is, in general, also not Hausdorff.

Let us finish this Lecture with three more examples of topological spaces:

*Example 13.* (The Cantor Set.) Consider the closed unit interval $[0, 1]$ and remove its middle third $(1/3, 2/3)$. For the remaining two closed intervals $[0, 1/3]$ and $[2/3, 1]$ remove their open middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ as well. For the remaining four closed intervals, repeat the procedure, and continue doing this forever. Computing the sum of the lengths of the removed open intervals, one finds that we have removed intervals of total length

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = 1\ !$$

Since we have removed the union of open sets from the closed set $[0, 1]$, the remaining set $K$ is closed, and is called the *Cantor* set. There are uncountably many points in $K$.

*Example 14.* (Pro-finite topology on $\mathbb{Z}$.) Consider a topology on the set of integer numbers, in which the open sets are the empty set and the unions of arithmetic progressions. In this topology, a single arithmetic progression is also a closed set.

*Example 15.* Consider the equivalence relation on $\mathbb{R}$ in which $x \sim y$ iff $x - y \in \mathbb{Z}$. The quotient space $\mathbb{R}/\sim$ is homeomorphic to the unit circle in $\mathbb{R}^2$ with its standard subspace topology.

## 2. CONNECTEDNESS.

It is intuitively clear, for example, that in the standard topology of the real line, a single open interval is connected, whereas the union of two disjoint open intervals is not. Note that in the subspace topology, in the latter case each one of the two disjoint open intervals is not only open, but also closed. This motivates the following general definition:

**Definition 2.1.** *A topological space $X$ is* connected*, if there is no proper subset of $X$ (i.e. different from $X$ and $\emptyset$), which is simultaneously open and closed in $X$.*

*Example 1.* If $X$ is a topological space with discrete topology, containing more than one element, then $X$ is disconnected, because each proper subset of $X$ is simultaneously open and closed.

*Example 2.* Let $X$ be a topological space, consisting of three elements, $a$, $b$, and $c$, such that open sets in $X$ are the empty set and those that contain the element $a$. This is an example of a connected topological space.

The following is straightforward:

**Lemma 2.2.** *Let $f : X \to Y$ be a continuous map between two topological spaces. If $X$ is connected, then $f(X)$ is connected as well.*

An extreme case of a disconnected topological space is formalized in the following definition.

**Definition 2.3.** *A topological space is* totally disconnected *if any subset containing more than one element is not connected.*

Of course, a discrete topological space is totally disconnected, but here is another example:

*Example 3.* The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected.

A reformulation of the definition of connectedness can be useful in some instances:

**Lemma 2.4.** *Let $Z = \{0, 1\}$ be a set of two elements with discrete topology. A topological space $X$ is connected, iff there is no continuous surjective map $X \to Z$.*

Using this result, we can prove:

**Lemma 2.5.** *Let $X$ and $Y$ be connected topological spaces, then their product $X \times Y$ is also connected.*

*Proof.* Assume, on the contrary, that there exists a non-constant map $f : X \times Y \to \{0, 1\}$. Assume that for a given point $(x_0, y_0) \in X \times Y$, we have $f(x_0, y_0) = 0$ and for another point $(x_1, y_1) \in X \times Y$, we have $f(x_1, y_1) = 1$. Let $f(x_0, y_1) = a$. Clearly, $a \neq 0$, since $X \hookrightarrow X \times Y$, given by $x \mapsto (x, y_1)$ is continuous, and therefore its composition with $f$ must be constant, since $X$ is connected. Similarly, one can show that $a \neq 1$, and get a contradiction. ○

Our next task is to classify all the connected subsets of $\mathbb{R}$.

**Definition 2.6.** *A subset $J$ of $\mathbb{R}$ is called an* interval*, if it contains more than one point, and $a, b \in J$, $a < c < b$, imply $c \in J$.*

There are nine types of intervals in $\mathbb{R}$: closed and open interval $(-\infty, +\infty)$, closed intervals $[a, b]$, $(-\infty, a]$, $[a, +\infty)$, open intervals $(a, b)$, $(-\infty, a)$, $(a, +\infty)$, and intervals that are neither open, nor closed: $(a, b]$ and $[a, b)$. In our convention, $a$ and $b$ are finite numbers, $a < b$.

**Theorem 2.7.** *The only non-empty connected subsets of $\mathbb{R}$ are the intervals and the single points.*

*Proof.* Suppose $A$ is a non-empty subset of $\mathbb{R}$, which is not an interval, and not a single point. Then we can find $a < c < b$ such that $a, b \in A$ and $c \notin A$. Consider two intersections, $(-\infty, c) \cap A$ and $(c, +\infty) \cap A$. They are both open in $A$ and complementary. Therefore, $A$ is not connected.

Next, suppose $A$ is not connected. Then $A$ can be written as $A = (F \cup G) \cap A$, where $F$ and $G$ are two closed, non-empty, non-intersecting subsets of $\mathbb{R}$.

Without loss of generality, let $a \in F \cap A$ and $b \in G \cap A$ be such that $a < b$. If $A$ were an interval, then we would have $[a, b] \subset A$.

The subset $[a, b] \cap G$ is closed in $\mathbb{R}$, so it contains its lower bound, $c$, such that $a < c$. Analogously, $[a, c] \cap F$ is closed in $\mathbb{R}$ and contains its upper bound, $d$, such that $d < c$. But then the non-empty open interval $(d, c)$ does not belong to neither $F$, nor $G$, so $A$ is not an interval. ○

One of the applications of this result is a well-known result from Calculus:

**Theorem 2.8** (Intermediate Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < f(b)$, then for each $y \in (f(a), f(b))$ there exists $c \in (a, b)$ such that $f(c) = y$.*

*Proof.* By Lemma 2.2, the image $f([a, b])$ is connected, therefore it is an interval. ○

**Theorem 2.9** (Fixed Point Theorem)**.** *Let $f : [0,1] \to [0,1]$ be a continuous map. There exists $c \in [0,1]$ such that $f(c) = c$.*

*Proof.* Consider the new continuous function $g(x) = f(x) - x$. Notice that $g(0) \geq 0$ and $g(1) \leq 0$. By the Intermediate Value Theorem, there exists $c \in [0,1]$ such that $g(c) = 0$, or $f(c) = c$. ○

**Definition 2.10.** *A subset $C \subset X$ is called a* connected component *of $X$ if it is connected and not properly contained in another connected subset of $X$.*

It can be readily shown that each connected component of $X$ is a closed subset and that $X$ is the union of its connected components.

*Example 4.* Consider the topological space $X = 0 \cup \{\frac{1}{n}\}$ , $n \in \mathbb{Z}_+$ with the topology induced from $\mathbb{R}$. Each point of $X$, except for 0, is both open and closed. However, the point $0 \in X$ is closed, but not open. Each point of $X$ is a connected component of $X$. This example shows that a connected component is not necessarily open.

**Definition 2.11.** *A* path *in a topological space $X$ is a continuous map $f : [0,1] \to X$. A space $X$ is called* path-connected*, if for any pair of points $x, y \in X$, there exists a path $f$ such that $f(0) = x$ and $f(1) = y$.*

**Lemma 2.12.** *If $X$ is path-connected, then $X$ is also connected.*

*Proof.* If $X$ is not connected, then we can represent $X = U \cup V$, where $U$ and $V$ are open, non-empty, and disjoint. Take $x \in U$ and $y \in V$ a consider a path $f$ connecting $x$ and $y$. Taking pre-images, we see that both $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty open subsets of $[0,1]$ which are disjoint and which together cover $[0,1]$. This contradicts the connectedness of the unit interval. ○

The converse to this Lemma is actually false:

*Example 5.* Consider the graph of the function $\sin(1/x)$ for $x \in (0,1]$, together with the interval $[-1,1]$ on the $y$-axis, and show that this is a connected subset of $\mathbb{R}^2$, which is not path-connected.

*Example 6.* Any convex subset of $\mathbb{R}^n$ is path-connected.

## 3. COMPACTNESS.

A *covering* of a topological space is a collection of its subsets, such that each point of $X$ belongs to at least one of those subsets. An *open covering* of $X$ is then a covering consisting of only open subsets.

**Definition 3.1.** *A topological space $X$ is said to be* compact*, if for each open covering $\{U_\alpha\}_{\alpha \in I}$, there is a finite subcovering $\{U_\beta\}_{\beta \in J}$, where $J$ is a finite subset of the indexing set $I$. A subset $Y \subset X$ is called compact, if it is a compact topological space in the subspace topology.*

**Proposition 3.2.** *The closed interval* $[0,1]$ *is compact.*

*Proof.* Assume there is a covering $\{U_\alpha\}$ of $[0,1]$ without a finite subcovering. Let us denote $a_0 = 0$ and $b_0 = 1$. One of the closed half-intervals, $[0,1/2]$ or $[1/2,1]$ then does not have a finite subcovering as well. Denote this new interval by $[a_1, b_1]$.

Continuing inductively, for each $i \in \mathbb{Z}_+$, and the interval $[a_i, b_i]$ at least one of its half-intervals, $[a_i, \frac{a_i + b_i}{2}]$ or $[\frac{a_i + b_i}{2}, b_i]$ does not have a finite subcovering. Call this new interval $[a_{i+1}, b_{i+1}]$.

Since the lengths of the intervals converges to zero, the sequences $\{a_i\}$ and $\{b_i\}$ converge to the same limit, $c$.

There exists an open set $V$ in the covering, containing $c$, together with its $\varepsilon$-neighbourhood, for some positive number $\varepsilon$. Therefore, $V$ also contains the whole interval $[a_N, b_N]$ for sufficiently large number $N$, which contradicts the fact that this interval does not have a finite subcovering. ○

*Example 1.* The open interval $(0,1)$ is not compact. Indeed, its open covering by the intervals $(\frac{1}{N}, 1)$ for $N \in \mathbb{Z}_+$ does not have a finite subcovering.

The following result follows directly from the definition (how?):

**Theorem 3.3.** *If $f : X \to Y$ is a continuous map and $A \subset X$ is compact, then $f(A)$ is compact as well.*

If we have a closed subset $C \subset X$ and an open covering $\{V_\alpha\}$ of $C$, such that $V_\alpha = U_\alpha \cap C$ and $U_\alpha$ is open in $X$, then we can add the open set $X \setminus C$ to the collection $\{U_\alpha\}$ to get an open covering of $X$. This leads to the following:

**Proposition 3.4.** *If $C$ is a closed subset of a compact topological space $X$, then $C$ is compact.*

Before stating the next result, we note the following direct consequence of definitions (prove it!):

**Lemma 3.5.** *Let $\mathcal{B}$ be a basis of topology for a topological space $Z$. If for each open covering of $Z$ by elements of $\mathcal{B}$ there exists a finite subcovering, then $Z$ is compact.*

Now we will establish the following important feature of compactness:

**Proposition 3.6.** *If $X$ and $Y$ are compact topological spaces, then their product $X \times Y$ is compact as well.*

*Proof.* Recall that a basis of topology for the product $X \times Y$ is given by $\{U \times V\}$, where $U$ is open in $X$ and $V$ is open in $Y$. Assume that we have an open covering $\{U_\alpha \times V_\alpha\}_{\alpha \in I}$ by elements of the basis. For each $x \in X$, let $I_x$ denote a finite subset of $I$, such that $(x \times Y)$ is covered by $\{U_\alpha \times V_\alpha\}_{\alpha \in I_x}$. This is possible to find, since $Y$ is compact. Denote:

$$U_x = \bigcap_{\alpha \in I_x} U_\alpha.$$

Clearly, the collection $\{U_x\}$ is an open covering of $X$, and since $X$ is compact, it admits a finite subcovering

$$U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_N}.$$

Now we claim that the finitely many open sets $\{U_\alpha \times V_\alpha\}$ for $\alpha \in I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_N}$ cover $X \times Y$.

Indeed, any point $(x, y) \in X \times Y$ belongs to $U_{x_j} \times Y$ for some $j$, which, in turn, is covered by $\{U_\alpha \times V_\alpha\}_{\alpha \in I_{x_j}}$. ○

Next, we will state (without proof) the Heine-Borel theorem.

**Theorem 3.7.** *A subset $C$ of $\mathbb{R}^n$ is compact iff $C$ is closed and bounded.*

One direction is easy: if $C$ is closed and bounded, then it is a closed subset of the compact cube $[-N, N]^n$ for sufficiently large size $N$, and thus compact as well.

Now, suppose we have a topological space $X$ and a subset $S \subset X$, which we assume to be infinite. A point $a \in X$ is said to be a *limit point* of $S$, if each of its neighbourhoods contains a point of $S$ different from $a$. We also say that $a$ is an *accumulation point* of $S$, if each of its neighbourhoods contains infinitely many points of $S$. For Hausdorff topological spaces these two properties are in fact equivalent. In addition, we say that $X$ has the Bolzano-Weierstrass property if each infinite subset of $X$ has a point of accumulation.

The main result for abstract topological spaces in this context is the following:

**Theorem 3.8.** *Let $X$ be a topological space with a countable basis of open sets. Then $X$ is compact iff it has the Bolzano-Weierstrass property.*

The proof of this result is based on the following two lemmas.

**Lemma 3.9.** *If $X$ has a countable basis of open sets, then each open covering of $X$ has a countable subcovering.*

*Proof.* Let $\mathcal{B} = \{B_i\}$ be such a basis and let $\{\mathcal{O}_\alpha\}$ be an open covering of $X$. For each point $x \in X$, there exists an open set $\mathcal{O}$ from the covering containing $x$. By definition of a basis, there exists $i$ such that $B_i \subset \mathcal{O}$. Choose one such set, $\mathcal{O}_i$, for each $i$. Naturally, $\{\mathcal{O}_i\}$ yields a countable subcovering. ○

**Lemma 3.10.** *Let $E$ be a subset of $X$, such that each infinite subset of $E$ has a point of accumulation. Then every countable covering of $E$ has a finite subcovering.*

*Proof.* Let $E \subset \bigcup \mathcal{O}_i$ be a countable collection of open sets, which induces a countable covering of $E$. Suppose that there is no finite subcovering. Then, for each $k \in \mathbb{Z}_+$, the open set

$$\Omega_k := \bigcup_{i=1}^{k} \mathcal{O}_i$$

does not cover $E$. Let us choose a point $x_k \in E \setminus \Omega_k$. This is an infinite subset, which by assumption has a point of accumulation $x_0$. But there exists a positive integer $m$ such that $x_0 \in \mathcal{O}_m$, since $\{\mathcal{O}_i\}$ is a covering. By definition of accumulation point, there exists $k > m$ such that $x_k \in \mathcal{O}_m \subset \Omega_k$, which contradicts the choice of $x_k$. ○

Conversely, if $E$ is compact, then it has the Bolzano-Weierstrass property. This can be seen using the argument from the preceding Lemma, where each point of $E$ would otherwise have a neighbourhood with only finitely many members of the sequence. Descending from the open covering to a finite subcovering, one would get a contradiction.

## REFERENCE.

I recommend the book by Bert Mendelson, *"Introduction to Topology"*, 3rd Edition, published by Dover, which is available from Amazon.com for < \$10.