Integration Workshop 2009
Calculus/Analysis notes

1 Calculus

1.1 Multivariate differential calculus

Definition 1 *Let $O \subset \mathbb{R}^n$ be open and $f : O \to \mathbb{R}^m$. Let $c \in O$ and let ϵ be small enough that $B(c, \epsilon) \subset O$. The function f is said to be differentiable at c if there is a linear function, called the total derivative, $T_c^f : \mathbb{R}^n \to \mathbb{R}^m$, such that*

$$f(c + v) = f(c) + T_c^f(v) + E_c(|v|) \tag{1}$$

for $|v| < \epsilon$, where the error term satisfies

$$\lim_{v \to 0} \frac{E_c(|v|)}{|v|} = 0 \tag{2}$$

We say f is differentiable on O if it is differentiable at every point in O. We say that f is continuously differentiable if T_c^f is a continuous function of c.

If the total derivative exists, then for all directions the directional derivative exists

$$D_v f(c) = \lim_{\epsilon \to 0} \frac{f(c + \epsilon v) - f(c)}{\epsilon}$$

and equals $T_c^f(v)$.

Let $e_1, e_2, \cdots, e_n$ be the standard basis for $\mathbb{R}^n$. Then the partial derivatives are the directional derivative in the directions of the standard basis:

$$\frac{\partial f}{\partial x_k} = D_{e_k} f$$

Note that both sides of this equation are vectors in $\mathbb{R}^m$. The components are

$$\frac{\partial f_j}{\partial x_k} = D_{e_k} f_j$$

The matrix representation of T in this basis is called the *Jacobian matrix*

$$Df(c) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(c) & \frac{\partial f_1}{\partial x_2}(c) & \cdots & \frac{\partial f_1}{\partial x_n}(c) \\ \frac{\partial f_2}{\partial x_1}(c) & \frac{\partial f_2}{\partial x_2}(c) & \cdots & \frac{\partial f_2}{\partial x_n}(c) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1}(c) & \frac{\partial f_m}{\partial x_2}(c) & \cdots & \frac{\partial f_m}{\partial x_n}(c) \end{pmatrix}$$

Theorem 1 *(Chain rule) Suppose that* $g : U \subset \mathbb{R}^k \to O$ *and* $f : O \subset \mathbb{R}^n \to \mathbb{R}^m$ *are differentiable at* $p \in U$ *and* $g(p) \in O$ *respectively. Then* $h = f \circ g$ *is differentable at* p *and*

$$T_p^h = T_{g(p)}^f \circ T_p^g$$

In matrix form

$$Dh(p) = Df(g(p))\, Dg(p)$$

Let $L(x_1, x_2) = \{\lambda x_1 + (1-\lambda)x_2 : 0 \le \lambda \le 1\}$ be the line segment connecting x_1 and x_2 in $\mathbb{R}^n$.

Theorem 2 *(Mean Value Theorem) Let* O *be an open subset of* $\mathbb{R}^n$ *and assume that* $f : O \to \mathbb{R}^m$ *is continuously differentiable on* O*. Choose* x_1 *and* x_2 *so that* $L(x_1, x_2) \subset O$*. Then for every vector* $a \in \mathbb{R}^m$*, there is a point* $c \in L(x_1, x_2)$ *such that*

$$a \cdot (f(x_2) - f(x_1)) = a \cdot T_c^f (x_2 - x_1)$$

We now consider higher order derivatives.

Theorem 3 *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$*. Then the following conditions are sufficient for the equality of the mixed partial derivatives*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(c) = \frac{\partial^2 f}{\partial x_j \partial x_i}(c)$$

1. Both $\partial f/\partial x_i and \partial f/\partial x_j$ *exist in an n-ball* $B(c, \delta)$ *and are differentiable at* c*.*

2. Both $\partial f/\partial x_i and \partial f/\partial x_j$ *exist in an n-ball* $B(c, \delta)$ *and* $\partial^2 f/\partial x_i \partial x_j$ *and* $\partial^2 f/\partial x_j \partial x_i$ *are both continuous at* c*.*

Call $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index if each of its entries are non-negative integers. Write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. This allows for the notational abbreviations

$$x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n}, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}$$

and provides for a compact notation for Taylors formula for functions f from $\mathbb{R}^n$ to $\mathbb{R}$. Write for $z, x \in \mathbb{R}^n$

$$f^{(k)}(a; x) = \sum_{\alpha : |\alpha| = k} D_\alpha f(a) x^\alpha$$

and assume that f and all of its partial derivatives of order up to $m-1$ are differentiable at each point of an open set $S \subset \mathbb{R}^n$. Choose x and a so that $L(a, x) \subset S$. Then for some $c \in L(a, x)$,

$$f(x) = f(a) + \sum_{k=1}^{m-1} \frac{1}{k!} f^{(k)}(a; x - a) + \frac{1}{m!} f^{(m)}(c; x - a)$$

1.2 Implicit functions

Let A be an $n \times n$ matrix. Then, for $y \in \mathbb{R}^n$, $Ax = y$ has a unique solution x whenever A has nonzero determinant. This suggests that in looking for a unique solution to $f(x) = y$, we consider the Jacobian determinant, the determinant of the Jacobian matrix,

$$J_f(x) = detDf(x) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$$

Theorem 4 *(Inverse function theorem) Let $f : S \to \mathbb{R}^n$ be continuously differentiable on $S \subset \mathbb{R}^n$. If the Jacobian determinant $J_f(a) \neq 0$ for some point $a \in S$, then there exists two open sets $X \subset S$ and $Y \subset f(S)$ and a uniquely determined function g defined on Y such that*

1. *$a \in X$ and $f(a) \in Y$*
2. *$Y = f(X)$*
3. *f is one-to-one on X*
4. *$g(Y) = X$*
5. *$g(f(x)) = x$ for every $x \in X$*
6. *g is continuously differentiable on Y .*

Note that for $y = f(x)$, $Dg(y)\,Df(x)$ is the identity linear transformation.

Theorem 5 *(Implicit function theorem) Let $S \subset \mathbb{R}^n \times \mathbb{R}^k$ and suppose that $f : S \to \mathbb{R}^n$ is continuously differentiable. Assume that $f(x_0, y_0) = 0$ and that the determinant of the $n \times n$ matrix $\partial f_j/\partial x_i(x_0, y_0)$ where i, j both range from 1 to n is not zero. Then there exists an open set $Y_0 \subset \mathbb{R}^k$ containing y_0 and a unique function $g : Y_0 \to \mathbb{R}^n$ such that*

1. *g is continuously differentiable*
2. *$g(y_0) = x_0$*
3. *$f(g(y), y) = 0$ for every $y \in Y_0$.*

1.3 Multivariable Riemann integrals

Definition of Riemann integrals on $\mathbb{R}^n$.

1.4 Change of vars in integrals

Let T be a one-to-one continuously differentiable mapping of an open set $V \subset \mathbb{R}^k$ into $\mathbb{R}^k$ such that the Jacobian determinant $J_T(x) \neq 0$ for all $x \in V$. Let f be a continuous function on $\mathbb{R}^k$ whose support is compact and lies in $T(V)$. Then

$$\int_{\mathbb{R}^k} f(y)\,dy = \int_{\mathbb{R}^k} f(T(x))\,|J_T(x)|\,dx$$

1.5 Differential forms and Stokes theorem

Let $K \subset \mathbb{R}^k$ be compact and let $V \subset \mathbb{R}^n$ be open. A k-surface is a continuously differentiable mapping $\Phi : K \to V$. For example, each component of a 1-surface is called a curve.

A *differential form of order* k, or briefly, a k-form, is a function ω, represented symbolically by

$$\omega = \sum a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

that assigns to each k surface Ψ in V a number

$$\int_\Phi \omega = \int_K \sum a_{i_1 \cdots i_k}(\Phi(u)) \frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(u_1, \cdots, u_k)} du$$

A 0-form is defined to be a continuous function of V . Integrals of 1-forms are called line integrals. Let $c \in \mathbb{R}$ and let $\omega, \omega_1, \omega_2$ be k-forms on V. Then

$$\int_\Phi c\omega = c \int_\Phi \omega$$

$$\int_\Phi (\omega_1 + \omega_2) = \int_\Phi \omega_1 + \int_\Phi \omega_2$$

For $\omega = a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and for $\bar{\omega}$ obtained from ω by interchanging some pair of subscripts, $\bar{\omega} = -\omega$.

Write the basic k-form $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, for $1 \le i_1 < \cdots < i_k$, giving the standard presentation

$$\omega = \sum_i a_I(x) dx_I$$

Differentiation of forms

The operator d is a mapping from k-forms to $(k+1)$-forms defined as follows:

1. For a class C^1 0-form f ,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

2. For the class C^1 k-form ω above in the standard presentation,

$$d\omega = \sum_I (da_I) \wedge dx_I$$

For i = 1, 2, let ω_i be class C^1 k_i-forms. Then

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge (d\omega_2)$$

If ω is of class C^2, $d(d\omega) = 0$.

k	m	Theorem
1	1	fundamental theorem
2	2	Green's theorem
3	3	divergence theorem
2	3	classical Stokes theorem

Definition 2 *A k-form ω is called exact if $\omega = d\eta$ for some $(k-1)$-form η. A class C^1 k-form is called closed if $d\omega = 0$.*

Every exact class C^1 form is closed. If the domain is a convex set, then the Poincare's lemma states that the converse is true.

Theorem 6 *(General Stokes' theorem) If Ψ is a k-chain of class C^2 in an open set $V \subset \mathbb{R}^m$ and if ω is a $(k-1)$-form of class C^1 in V, then*

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega$$

Various theorems from calculus and vector calculus are special cases of this general theorem as indicated in the table.

Theorem 7 *(Green's Theorem) Let C be a simple closed curve in the xy-plane Let $M(x,y)$ and $N(x,y)$ be continuously differentiable on an open set containing C and the region it encloses. Let R be the region enclosed by C. Then*

$$\int_C (Mdx + Ndy) = \int_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dxdy$$

Theorem 8 *(Divergence Theorem) Let F be a continuously differentiable vector field on an open set $V \subset \mathbb{R}^3$, and let $C \subset V$ be closed with positively oriented boundary ∂C. Then*

$$\int_C (\nabla \cdot F) dV = \int_{\partial C} (F \cdot n)\, dA$$

where n is a unit normal vector, pointing outwards.

Theorem 9 *(classical Stokes Theorem) Let F be a continuously differentiable vector field on an open set $V \subset \mathbb{R}^3$, and let $S \subset V$ be a 2-surface of class C^2. Then*

$$\int_S (\nabla \times F) \cdot n\, dV = \int_{\partial S} (F \cdot t)\, ds$$

where t is a oriented unit tangent vector.

2 Real Analysis

2.1 Sequences and series

Definition 3 *Let a_n be a sequence of real or complex numbers. We say that a_n converges to a if for every $\epsilon > 0$ there is an N such that $n \geq N$ implies $|a_n - a| < \epsilon$. We say that a_n is a Cauchy sequence if for every $\epsilon > 0$ there is an N such that $n, m \geq N$ implies $|a_n - a_m| < \epsilon$.*

By the completeness axiom of the real numbers, a monotone sequence converges if and only if it is bounded. Given a sequence a_n, define $b_n = \sup\{a_k : k \geq n\}$. Then b_n is a nonincreasing sequence and so has a limit. Call this the limit superior or just lim sup a_n and write $\limsup_{n\to\infty} a_n$. Similarly, the limit inferior or lim inf is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

The lim inf and lim sup always exist although the $liminf$ can be $-\infty$ and the lim sup can be $+\infty$. We always have $\liminf a_n \leq \limsup a_n$. They are equal if and only if a_n converges. In this case they are equal to the limit of a_n.

Convergence tests for series and sequences:

1. *Integral test*: Let f be a positive decreasing function defined on $[1, \infty)$ such that $lim_{x\to\infty} f(x) = 0$. For $n = 1, 2, \cdots$, define

$$s_n = \sum_{k=1}^{n} f(k), \quad t_n = \int_1^n f(x)\, dx,$$

Then s_n converges if and only if t_n converges.

2. *Ratio and root tests*: Given a series $\sum_{n=1}^{\infty} a_n$ of nonzero complex terms, let

$$r_- = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \quad r_+ = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \quad \rho = \limsup_{n\to\infty} |a_n|^{1/n}$$

(a) The series converges absolutely if either $r_+ < 1$ or $\rho < 1$.
(b) The series diverges if either $r_- > 1$ or $\rho > 1$.
(c) In all other cases the tests are inconclusive.

3. *Dirichlet's test*: Given a series $\sum_{n=1}^{\infty} a_n$ of nonzero complex terms such that the partial sums are a bounded sequence. Let b_n be a decreasing sequence which converges to 0. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

4. *Abels test:* The series $\sum_{n=1}^{\infty} a_n b_n$ converges if $\sum_{n=1}^{\infty} a_n$ converges and b_n is monotone and bounded.

2.2 Infinite products

Let u_n be a sequence of complex numbers. The infinite product $\prod_{n=1}^{\infty} u_n$ is said to converge if there is an N such that $u_n \neq 0$ for $n \geq N$ and the sequence $p_k = \prod_{n=N+1}^{k}$ has a nonzero limit p as $k \to \infty$. In the case of convergence, $\prod_{n=1}^{\infty} u_n$ is defined to be $pp_1 \cdots p_N$.

There is a connection betweeen convergence of sums and of products. For $a_n > 0$, the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

We say that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges. Absolute convergence of the infinite product implies convergence of the product.

2.3 Sequences of functions

Continuity and uniform continuity

A function f is continuous at a if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. A function f is uniformly continuous on S if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $a \in S$ and all x with $|x-a| < \delta$, we have $|f(x)-f(a)| < \epsilon$.

If S is compact and f is continuous on S then it is uniformly continuous on S.

Convergence, uniform convergence and continuity

A sequence of functions is said to *converge pointwise* to a limit function f on a set S provided that for every $x \in S$, and each $\epsilon > 0$, there exists N, depending possibly on both x and ϵ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$.

If the choice of N does not depend on x, the sequence of functions is said to *converge uniformly.*

If $f_n \to f$ uniformly on S and each f_n is continuous at a point c, then f is continuous at c. Let f_n be a sequence of functions defined on a set S. For each $x \in S$, set

$$s_n(x) = \sum_{k=1}^{n} f_n(x)$$

If $s_n \to s$ uniformly on S, then we say that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.

Theorem 10 *(Weierstrass M-test) Let M_n be a sequence of nonnegative numbers such that $0 \leq |f_n(x)| \leq M_n$ for $n = 1, 2, \cdots$ and every $x \in S$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.*

The L^{∞} norm

Consider the vector space $C(S)$, the real valued continuous functions on S, and define the infinity norm (or sup norm) by

$$||f||_{\infty} = \sup_{x \in S} |f(s)|$$

Then $|| \quad ||_\infty$ is a norm, meaning that

1) $||f||_\infty \geq 0$ and $||f||_\infty = 0$ if and only if $f(x) = 0$ for all $x \in S$.

2) $||af||_\infty = |a|||f||_\infty$ for every $a \in \mathbb{R}$.

3) $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$

This norm induces a metric $d(f, g) = ||f - g||_\infty$. The theorems above on uniform continuity show that $C(S)$ with this metric is a complete metric space.

Integration and differentiation

Many of the theorems on uniform convergence permit the reversal of the order of taking of limits.

Theorem 11 *(Integration) Let $\alpha(x)$ have bounded variation on $[a, b]$ and assume that f_n is a Riemann integrable function for $n = 1, 2, \cdots$. Define*

$$g_n(x) = \int_a^x f_n(t) d\alpha(t), \quad x \in [a, b]$$

Assume there exists f so that $d(f_n, f) \to 0$. Then

(a) f is Riemann integrable with respect to α and

(b) $d(g_n, g) \to 0$ where

$$g(x) = \int_a^x f(t) d\alpha(t), \quad x \in [a, b]$$

Theorem 12 *(Differentiation) Assume that f_n is differentiable on (a, b) and that there exist a function g so that $d(f_n', g) \to 0$ and a point $c \in (a, b)$ so that $f_n(c)$ converges. Then*

(a) there exists f so that $d(f_n, f) \to 0$, and

(b) f is differentiable with derivative g.

3 Complex Analysis

3.1 Analytic functions

Let O be an open subset of $\mathbb{C}$. Let $f : O \to \mathbb{C}$. We say f is analytic at z_0 if the following complex limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

One can think of a function from $\mathbb{C}$ to $\mathbb{C}$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. We write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Then the function is $F(x, y) = (u(x, y), v(x, y))$. It is important to understand that analyticity is a much stronger property than requiring that F have a total derivative. The above limit involves complex numbers and so it includes as special cases z approaching z_0 along any direction. These "directional limits" must all give the same complex number as the limit. In particular, by considering taking the limit in the coordinate directions, one obtains the Cauchy Riemann equations.

Theorem 13 *f is analytic at z_0 if and only if the total derivative of F exists at (x_0, y_0) and*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

3.2 Power series

An infinite series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series centered at z_0. Define r by $1/r = \limsup_{r \to \infty} |a_n|^{1/n}$ (We make the conventions that $1/0 = \infty$ and $1/\infty = 0$.) Then by the root test, the series converges absolutely if $|z - z_0| < r$ and diverges if $|z - z_0| > r$. Furthermore:

1. The series converges uniformly on every compact subset of $B(z_0, r)$.
2. The function f can be differentiated term by term for any $z \in B(z_0, r)$,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

3. The power series for f' has radius of convergence r.
4. Repeated differentiation and evaluation of this yields $a_k = f^{(k)}(z_0)/k!$.

Theorem 14 *Suppose that the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has a nonzero radius r of convergence. Then $f(z)$ is analytic on $B(z_0, r)$. Conversely, if $f(z)$ is analytic at z_0, then there is a power series with a nonzero radius of convergence that converges to f in a neighborhood of z_0.

3.3 Integration

Difference between complex contour integrals and line integrals in $\mathbb{R}^2$.

Definition 4 *A domain D is simply connected if the region bounded by every simple closed curve in D is contained in D, i.e., every simple closed curve in D may be continuously contracted to a point without leaving D.*

Theorem 15 *(one of many "Cauchy's theorem") If D is a simply connected open set and f is analytic on D and γ is a differentiable closed curve in D, then*

$$\oint_{\gamma} f(z) dz = 0 \tag{3}$$

3.4 Zeroes, Poles and residues

We say f has a zero at z_0 if $f(z_0) = 0$. In this case it is possible to write it in the form $f(z) = (z - z_0)^n g(z)$ in a neighbor of z_0 where g does not vanish on this neighborhood. The integer n is unique and called the *order* of the zero.

A neigborhood of a point z_0 means an open set containing z_0. By a deleted neighborhood of z_0 we will mean a neighborhood of z_0 with z_0 removed. f has an isolated singularity at z_0 if it is analytic on a deleted neighborhood of z_0.

If f has an isolated singularity at z_0, and we can redefine it at z_0 so that the function is analytic at z_0, then we say f has a removable singularity at z_0. Otherwise we consider $1/f$ where $1/f$ is defined to be 0 at z_0. It this is anayltic at z_0 we say f has a pole at z_0. The order of the pole is the order of the zero of $1/f$ at z_0. If it does not have a pole we say it has an essential singularity.

Theorem 16 *If f has a pole of order n at z_0 then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + g(z)$$

where $g(z)$ is analytic at z_0.

The *principal part* of $f(z)$ (at z_0) is

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}$$

The *residue* of f at z_0 is a_{-1}

4 Differential Equations

4.1 First order ordinary differential equations

Let $V \subset \mathbb{R}^n$ and $I = [t_0, t_f]$ and $\phi : I \times V \to \mathbb{R}^n$. A solution to the initial value problem $y' = \phi(t, y), y(t_0) = y_0$ is a differentiable function f on I such that $f(t_0) = y_0$, $f(t) \in V$ and $f'(t) = \phi(t, f(t))$. The ODE is called autonomous if ϕ is independent of t. The basic estimate used to study the dependence of solutions on initial conditions is Gronwalls inequality.

Theorem 17 *(Gronwall's inequality) Let $f, g : [a, b) \to \mathbb{R}$ be continuous and nonnegative. Suppose*

$$f(t) \le K + \int_a^t f(s)g(s)\,ds \quad K \ge 0$$

Then

$$f(t) \leq Kexp\left(\int_a^t g(s)\,ds\right)$$

for $t \in [a, b)$.

4.2 Linear ODE's

A *linear system* is one in which $\phi(t, y) = A(t)y + g(t)$. It is called *homogeneous* if $g(t) = 0$ for all t. For a homogeneous autonomous linear system $y' = Ay$, a solution is $y(t) = exp(t - t_0)Ay_0$. By Gronwalls inequality, this solution is unique.

For nonautomonous systems having continuous A, the solutions of $y' = A(t)y$ form a vector space of dimension n over the complex numbers. An $n \times n$ matrix whose columns are linearly independent solutions is called a fundamental matrix. Once this matrix valued function has been found, we can find a solution to the non-homogeneous system by variation of constants.

Theorem 18 *If* Φ *is a fundamental matrix of* $y' = A(t)y$ *on* I*, then the function*

$$\psi(t) = \Phi(t) \int_{t_0}^t \Phi^{-1}(s)g(s)\,ds$$

is the unique solution of the nonhomogenous linear system above with initial condition $\psi(t_0) = 0$.

4.3 Existence of solutions and iteration techniques

Lemma 1 *Let* ϕ *be Lipschitz in* y *uniformly in* t *and assume that* $B(y_0, r) \subset V$ *. Choose* M *so that* $|\phi(t, y)| \leq M$ *for* $(t, y) \in I \times B(y_0, r)$*. Set* $t_0 \in I$ *and* $\delta = r/M$ *. Then there is a unique* C^1 *function* $y(t)$*,* $t \in (t_0 - \delta, t_0 + \delta)$ *satisfying* $y(t) \in B(y_0, r)$ *that is a solution to the ordinary differential equation.*

The differential equation is equivalent to

$$y(t) = y_0 + \int_{t_0}^t \phi(t, y(s))\,ds$$

The *Picard iteration technique* begins by seting $y_0(t) = y_0$ and defining inductively

$$y_{n+1}(t) = y_0 + \int_{t_0}^t \phi(t, y_n(s))\,ds$$

Let K be the Lipschitz constant. By induction, we find that

$$|y_{n+1}(t) - y_n(t)| \leq \frac{MK^n|t - t_0|^{n+1}}{(n+1)!}$$

Thus, y_n is a Cauchy sequence in the L^∞ norm. Consequently, the limit y is a continuous curve that is a solution. To check uniqueness, let $\hat{y}$ be another solution. Then check that

$$|y_n(t) - \hat{y}(t)| \leq \frac{MK^n|t - t_0|^{n+1}}{(n+1)!}$$

to see that $\hat{y} = y$.