

Integration Workshop 2009

Calculus/Analysis Problems

1 Calculus

1. $f : O \rightarrow \mathbb{R}^m$ is differentiable and $Df = 0$ on O , an open subset of \mathbb{R}^n . Show that f is a constant on O .
2. Let f and $\frac{\partial f}{\partial y}$ be continuous on $[0, 1] \times [0, 1]$ and assume that $p, q : [0, 1] \rightarrow [0, 1]$ are differentiable. Define

$$F(y) = \int_{p(y)}^{q(y)} f(x, y) dx, \quad y \in [0, 1]$$

Use the chain rule to find $F'(y)$. Hint: Consider $G(x_1, x_2, x_3) = \int_{x_1}^{x_2} f(t, x_3) dt$.

3. Evaluate the derivatives of the following matrix functions:
 - (a) $\text{inv} : GL(n) \rightarrow GL(n)$ given by $\text{inv}(M) = M^{-1}$.
 - (b) The determinant function which maps $GL(n)$ to \mathbb{R} .
4. (a) Give an example of a function of two variables that is discontinuous at the origin, but whose partial derivatives at the origin exist.
(b) Give an example of a function of two variables all of whose directional derivatives exist at the origin, but the function itself is not differentiable at the origin.
5. Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$. We say f is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, for $y, y_1, y_2 \in \mathbb{R}^m$ and $a \in \mathbb{R}$ we have

$$\begin{aligned} f(ax, y) &= af(x, y) = f(x, ay) \\ f(x_1 + x_2, y) &= f(x_1, y) + f(x_2, y) \\ f(x, y_1 + y_2) &= f(x, y_1) + f(x, y_2) \end{aligned}$$

Prove that if f is bilinear then f is differentiable and express its derivative in terms of f .

6. Show that every point p on the sphere $x^2 + y^2 + z^2 = 1$ has a (3 dimensional) neighborhood U such that there is a smooth, one to one mapping of an open neighborhood V of the origin in \mathbb{R}^3 such that the plane $z = 0$ maps to the surface of the sphere.
7. Let F denote the vector field $(x^2 - z^2, 2xy, z)$ on \mathbb{R}^3 . Compute in two different ways the surface integral

$$\int \int_T F \cdot n dA$$

where T denotes the surface of the tetrahedron bounded by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$ and n denotes the outward normal to T . Use the two answers to verify the divergence theorem.

8. Compute in two different ways the line integral

$$\oint ydx - xdy + z^2dz$$

where C is the intersection of the paraboloid $z = x^2 + 4y^2$ with the cylinder $x^2 + y^2 = 9$, traversed counter-clockwise when viewed from the point $(0, 0, 100)$. Use the two answers to verify Stokes' theorem

9. Let $f(x, y) = (x - y)/(x + y)^2$. Show that

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right) dy = -1/2, \quad \text{but} \quad \int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = 1/2$$

10. Show that the systems of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z ; for x, z, u in terms of y ; for y, z, u in terms of x ; but not for x, y, z in terms of u .

11. Let $U \subset \mathbb{R}^3$ be open, f a differentiable function on U and F a differentiable vector field on U . Define the forms

$$\begin{aligned} \omega_F^1 &= F^1 dx + F^2 dy + F^3 dz \\ \omega_F^2 &= F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy \end{aligned}$$

(a) Show that

$$\begin{aligned} df &= \omega_{\nabla f}^1, \\ d\omega_F^1 &= \omega_{\nabla \times F}^2 \\ d\omega_F^2 &= (\nabla \cdot F) dx \wedge dy \wedge dz \end{aligned}$$

(b) Use (a) to show how Green's theorem, the divergence theorem and the classical Stoke's theorem are all special cases of the general Stoke's theorem.

(c) Use (a) to show that

$$\begin{aligned}\nabla \times (\nabla f) &= 0 \\ \nabla \cdot (\nabla \times F) &= 0\end{aligned}$$

(d) Use (c) to show that if F is a vector field on a convex open set U and $\nabla \times F = 0$, then there is a differentiable function f on U such that $F = \nabla f$. Similarly, if $\nabla \cdot F = 0$ on U , show that there is a vector field G on U with $F = \nabla \times G$.

2 Real Analysis

12. Assume that $a_n > 0$ and $b_n > 0$ for $n = 1, 2, \dots$, and suppose that $\lim_{n \rightarrow \infty} a_n/b_n = 1$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

13. Let a_n be a real sequence. Let A be the set of all real numbers a such that there is a subsequence of a_n which converges to a . Prove that

$$\sup A = \limsup_{n \rightarrow \infty} a_n$$

14. Let f_n be uniformly continuous on S , f continuous on S and suppose f_n converges uniformly to f . Does it follow that f is uniformly continuous? Either prove that it does or give a counter example.

15. Let f be a continuous real valued function on $[0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x)$ exists (and is finite). Prove that f is uniformly continuous on $[0, \infty)$.

16. Given a set S in a metric space, let \bar{S} denote the set obtained by adding all the limit points of S to S , i.e., $p \in \bar{S}$ if there is a sequence $x_n \in S$ such that $x_n \rightarrow p$. Show that $\underline{\bar{S}} = \bar{S}$.

17. Show that $\mathbb{Q}[x]$, the set of all polynomials in one variable with rational coefficients is countable.

18. (a) Give an example of a function $f : [0; 1] \rightarrow \mathbb{R}$ that is continuous at all the irrationals and discontinuous at all the rationals.

(b) Show that the set of points of discontinuity of an arbitrary real valued function is always a F_σ set and the set of points of continuity is always a G_δ set.

(c) [Optional.. hard..] There is no real valued function that is continuous on all the rationals but discontinuous on all the irrationals (Use Baire category theorem).

19. (a) Let f be a continuous function on $[0, 1]$. Show that the following limits exist and evaluate them:

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx$$

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx$$

(b) Let g be a differentiable function on $[0, 1]$ such that $g(1) = 0$. Show that the following limit exists and evaluate:

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 x^n g(x) dx$$

20. Give a counter example to the following statement: A sequence of differentiable functions f_n converges uniformly to a differentiable function f . This implies that the sequence of derivatives f'_n converges pointwise to f' .

21. f_n is a sequence of uniformly bounded non-negative Riemann integrable functions which are monotone non-decreasing on $[0, 1]$, i.e $n \geq m$ and $x \in [0, 1]$ implies that $f_n(x) \geq f_m(x)$, and $0 \leq f_n(x) \leq K < \infty$ for all n, x .

(a) Show that the sequence $f_n(x)$ converges pointwise to a bounded function $f(x)$.

(b) Show that the sequence of real numbers $s_k = \int_0^1 f_k(x) dx$ also converges to some number s .

(c) Is it true that f is Riemann integrable, and $\int_0^1 f(x) dx = s$? Prove or give a counter example.

22. Let \mathcal{F} be the set of continuous functions on S which have a continuous derivative on S . Define a metric d on this set by

$$d(f, g) = \|f - g\|_\infty + \|f' - g'\|_\infty$$

Is this a complete metric space?

3 Complex Analysis

23. Let $f(z)$ be analytic at c . Write $f'(c) = re^{i\theta}$. Write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. We can think of f as a map $(u(x, y), v(x, y))$ from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. The total derivative of this map at c is a two by two matrix. Find it in terms of r and θ . Express the direction derivatives of the map on \mathbb{R}^2 in terms of r and θ .

24. Define

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k$$

(a) If the radius of convergence of the above power series is r , then show that the radii of convergence for the series obtained by differentiating and integrating the above series termwise is also r .

(b) Show that the termwise differentiated series converges uniformly on every disk of the form $|z - z_0| \leq \rho < r$. Use this to show that $f(z)$ is given by termwise differentiating the above series, for any compact subset of the disk $|z - z_0| < r$.

25. Suppose that γ is a piecewise smooth positively, counterclockwise oriented, simple closed curve. Use Green's Theorem to show that the value of the integral

$$\oint_{\gamma} \frac{dz}{z - p}$$

equals 0 if p is outside γ and $2\pi i$ if p is inside γ .

26. If $f(z)$ is analytic on the closed disk $B(z_0, r)$, show that there is a constant C such that the derivatives of f at z_0 can be bounded by

$$f^{(n)}(z_0) \leq \frac{Cn!}{r^{n+1}}$$

27. Let $f(z)$ be analytic at z_0 . Prove that for sufficiently small ϵ ,

$$\frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} \frac{f(z)}{(z-z_0)^n} dz = \frac{f^{(n-1)}(z_0)}{(n-1)!}$$

The contour is the circle centered at z_0 with radius ϵ traversed in the counter-clockwise direction. This is a standard theorem in complex analysis books. The exercise is to prove it directly from the statement of Cauchy's theorem given in the notes. Hint: power series.

28. For each of the following functions f and z_0 , determine if the function has a pole or essential singularity at z_0 . In the case of a pole determine the order of the pole and the principal part.

(a) $f(z) = \frac{1}{z \sin(z)}$, $z_0 = 0$

(b) $f(z) = \frac{1}{(z^2+1)^2}$, $z_0 = i$

(c) $f(z) = \exp(-1/z)$, $z_0 = 0$.

(d) $f(z) = \tan^2(z)$, $z_0 = \pi/2$.

29. A Möbius transformation, or a fractional linear transformation, is a mapping of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$. This can be extended to the Riemann sphere $= \mathbb{C} \cup \{\infty\}$

$$f(\infty) = a/c, \quad f(-d/c) = \infty$$

- (a) Show that the Möbius transformations form a group.
- (b) Show that the Möbius transformations map circles to circles on the Riemann sphere (Note: A straight line on \mathbb{C} is considered a circle through ∞ on the Riemann sphere).

4 Differential Equations

30. Derive a series representation for the fundamental matrix $\Phi(t)$ of the linear system $y' = Ay$ using Picard iteration.

31. Solve $y' = Ay + g(t)$ with $y(0) = (0, 0)$, and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 2e^{-t} \\ te^{-2t} \end{pmatrix}$$

32. Let J_0 be an $n \times n$ matrix that has the value 1 above the diagonal and 0 elsewhere.

- (a) Find $\exp(tJ_0)$.
- (b) Verify that the identity matrix I and J_0 commute.
- (c) Use this to find $\exp(t(\lambda I + J_0))$.
- (d) Describe the unique solution to the differential equation $y' = Ay$ where A is a matrix in Jordan canonical form.