Integration Workshop 2009

Linear Algebra Part I: Dinesh Thakur

Lecture plan:

We will follow roughly the same lecture plan from the last year (distributed separately). I will give the first two lectures covering vector spaces, bases, linear transformations, duality, eigenvalues, determinants, canonical forms and Andrea Young will give the third lecture covering bilinear forms. See the plan for more details.

Projects

The webpage has a long list of projects (some of which would be new) from which you can choose.

Problems

Problem list from previous years is being distributed separately. Those are important problems and you should (eventually) try all. Here are some additional problems.

Additional Linear Algebra Problems

1. Arrange the following Q-vector spaces in the isomorphism classes giving justification. (i) R, (ii) C, (iii) $V := Q[x]$, (iv) V^*, (v) V^{**}, (vi) Direct sum of a countably many copies of Q, (vii) Direct product of a countably many copies of Q.

2. Recall Vandermonde determinant identity (2.4 from 2007/2008 workshop problems) $|x_j^{i-1}| = \prod_{i>j}(x-i-x-j)$. Derive similarly its F_q-lineralized version (Moore identity)

$$|x_j^{q^{i-1}}| = \prod_i \prod_{f_j \in F_q} (x_i + f_{i-1}x_{i-1} + \cdots + f_1 x_1)$$

Note that if we choose $x_j = t^{j-1}$ the resulting determinant is both Vandermonde and Moore type. Deduce that the product of all monic polynomials in $F_q[t]$ of degree n is

$$(t^{q^n} - t)(t^{q^n} - t^q) \cdots (t^{q^n} - t^{q^{n-1}}).$$

3. Let

$$A = \begin{pmatrix} 13/9 & -10/9 \\ -5/9 & 8/9 \end{pmatrix}, \quad v = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

Show that the difference between $A^n v$ and transpose of $(4 * 2^n, -2 * 2^n)$ tends to zero, exponentially fast, as n tends to infinity. Discuss how you can estimate similarly $A^n v$ for a square complex matrix A of size n and a complex vector v of size n under suitable conditions.

4. Solve the simultaneous equations

$$x^3 + 2x^2 y + 2y(y-2)x + y^2 - 4 = 0, \quad x^2 + 2xy + 2y^2 - 5y + 2 = 0$$

by eliminating y from these two quadratics in y using the resultant determinants.

5. Let $\sum_0^a a_i \alpha^i = 0$ and $\sum_0^b b_i \beta^i = 0$, with say $a_i, b_i \in Q, \alpha, \beta \in C$. Show how you can write down explicit (in terms of a_i, b_i) polynomial equations (of degree at most ab) with rational coefficients satisfied by $\alpha + \beta$, $\alpha\beta$. (Hint: Consider span of $v_{ij} = \alpha^i \beta^j$, with $0 \leq i < a, 0 \leq j < b$ and multiplication action of $\alpha + \beta$ or $\alpha\beta$ on it. Use that non-zero kernel for a linear transformation makes its determinant zero.)