INTEGRATION WORKSHOP 2007
LINEAR ALGEBRA EXERCISES

These exercises are adapted from exercises written by Douglas Ulmer and Nick Rogers from previous Integration Workshops.

1. Vector Spaces and Linear Transformations

Unless otherwise specified, assume all vector spaces are finite dimensional.

1.1. Suppose that $T_1 : V \to W$ and $T_2 : V \to X$ are linear transformations, and choose bases $\{v_i\}$ and $\{w_j\}$ for V and W, respectively. Let A_1 and A_2 be the matrices for T_1 and T_2, respectively, with respect to these bases. Show that the matrix for $T_2 \circ T_1 : V \to X$ with respect to the basis $\{v_i\}$ is $A_2 A_1$. Conclude that matrix multiplication is associative.

1.2. Let U_1 and U_2 be subspaces of a vector space V. Prove that

$$\dim(U_1) + \dim(U_2) = \dim(U_1 \cap U_2) + \dim(U_1 + U_2).$$

1.3. A system of linear equations may have no solutions, a unique solution, or many solutions. Explain this in terms of the image and kernel of a linear transformation.

1.4. Let V and W be vector spaces, and let $\mathrm{Hom}(V, W)$ be the set of linear transformations from V to W. Prove that $\mathrm{Hom}(V, W)$ has a natural vector space structure.

Let $e_1, \ldots, e_m$, and $f_1, \ldots, f_n$, be bases for V and W, respectively, and let $\ell_1, \ldots, \ell_m$, be the dual basis for V^*. For $1 \le i \le m$ and $1 \le j \le n$, prove that the formula

$$E_{ij}(v) = \ell_j(v) f_i,$$

defines an element $E_{ij} \in \mathrm{Hom}(V, W)$. Furthermore, show that these elements define a basis for $\mathrm{Hom}(V, W)$, and conclude that $\dim(\mathrm{Hom}(V, W)) = \dim(V)\dim(W)$.

If you know about tensor products, use this to show that $\mathrm{Hom}(V, W) \cong V^* \otimes W$. To what extent does all of this work if we allow infinite dimensional vector spaces?

1.5. An element $T \in \mathrm{Hom}(V, V) = \mathrm{End}(V)$ is called *nilpotent* if there is some positive integer n such that $T^n = 0$. Prove that T is nilpotent if and only if there is some basis for V such that the matrix for T is strictly upper triangular (zeroes on and below the main diagonal). Conclude that if T is nilpotent, then $T^{\dim(V)} = 0$.

1.6. An element $T \in \mathrm{Hom}(V, V) = \mathrm{End}(V)$ is called *semisimple* if there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that $V \cong \oplus_{i=1}^n V_i$ and $T(V_i) \subset V_i$ for each i. (If the field of scalars is not algebraically closed, then this is actually a non-standard definition for semisimple). Prove that T is semisimple if and only if there exists a basis for V such that the matrix for T is diagonal.

1.7. Let $T \in \mathrm{Hom}(V, V) = \mathrm{End}(V)$, where $\dim(V) = n$. Prove that there exists a basis of V such that the matrix for T is upper triangular if and only if there subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim(V_i) = i$ and $T(V_i) \subset V_i$.

Prove that if V is a $\mathbb{C}$-vector space, then T has an eigenvector. (Hint: Let v be any non-zero vector, and consider the $n+1$ vectors $v, Tv, T^2v, \ldots, T^nv$, which must be linearly dependent).

Prove by induction on the dimension of V that the conditions in the first part of the problem are always satisfied in the case that V is a $\mathbb{C}$-vector space.

1.8. If $T : V \to W$ is a linear transformation and $T^* : W^* \to V^*$ is its transpose, prove that the kernel of T is the orthogonal complement of the image of T^*, in other words,

$$\ker(T) = \{v \in V \mid \ell(v) = 0 \text{ for all } \ell \in \mathrm{Im}(T^*)\}.$$

Similarly, show that the image of T is the orthogonal complement of the kernel of T^*.

1.9. Prove that if $W \subset V$ is a subset with $\dim(W) = \dim(V) - 1$ (that is, W has *codimension* 1 in V), then there exists an element $\ell \in V^*$ such that W is the kernel of ℓ. Generalize to an arbitrary subspace $W \subset V$.

1.10. Suppose that the system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, \ldots, m,$$

has real coefficients, so that $a_{ij}, b_i \in \mathbb{R}$ for all i, j. Prove that if the system has a solution with $(x_j) \in \mathbb{C}^n$, then it has a solution with $(x_j) \in \mathbb{R}^n$. In this case, is every solution in $\mathbb{R}^n$?

1.11. Use Gaussian elimination to show that an arbitrary matrix can be written as a product PLU, where P is a (square) permutation matrix, L is a (square) lower triangular matrix, and U is a (not necessarily square) upper triangular matrix.

2. Eigenvalues and Jordan form

For these problems, you may assume all vector spaces are finite dimensional and over the complex numbers

2.1. Describe the sizes of the Jordan blocks of $T \in \mathrm{End}(V)$ in terms of the dimensions of $\ker((T - \lambda)^i)$, where λ is an eigenvalue of T and $i = 1, 2, 3, \ldots$.

2.2. Give necessary and sufficient conditions for an endomorphism $T : V \to V$ to have a square root, that is, for the existence of an endomorphism $U : V \to V$ such that $U^2 = T$.

2.3. Prove that any $n \times n$ matrix over $\mathbb{C}$ is conjugate to its transpose.

2.4. The Vandermonde determinant of order n is the determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Show that
$$V_n = \prod_{1 \le i < j \le n} (x_j - x_i).$$
(Hint: First show that $x_j - x_i$ divides V_n for all $i < j$.)

2.5. Prove that the set of diagonalizable $n \times n$ matrices is dense in $\mathbb{C}^{n^2}$ in the following sense: Given A, for every $\varepsilon > 0$, there is a B all of whose entries are $< \varepsilon$ such that $A + B$ is diagonalizable. (Hint: First show this for Jordan blocks.)

2.6. Prove that if T is an endomorphism such that $T^m = I$ for some integer $m \geq 1$, then T is diagonalizable. Use your proof to give a more general condition for T to be diagonalizable.

2.7. Let $T \in \mathrm{End}(V)$. For $v \in V$, the *cyclic subspace* generated by v is the span of $v, T(v), T^2(v), \ldots$. Prove that the cyclic subspace generated by v is one dimensional if and only if v is an eigenvector of T. Give a more general statement for the dimension for various v, given the Jordan form of T.

2.8. (Uniqueness of Jordan decomposition) Let $T \in \mathrm{End}(V)$, and suppose that $T = S+N = S'+N'$ are two Jordan decompositions of T. Prove that $SS' = S'S$ and $TT' = T'T$.

Prove that $S - S'$ is semisimple and $T - T'$ is nilpotent (it is necessary to use the first part of the problem here, otherwise, it is not necessarily true). Conclude that $S = S'$ and $T = T'$.

2.9. Determine the set of matrices which commute with a given matrix. Start with the cases diagonal matrices with distinct eigenvalues, diagonal matrices with repeated eigenvalues, a Jordan block, and so on.

3. Bilinear Forms

3.1. Use the Gram-Schmidt process to prove that every invertible complex matrix can be written as the product of a unitary matrix and an upper triangular matrix (this is called the *QR decomposition*. What is the version over the real numbers?

3.2. Prove that an invertible real matrix A can be written as QP where P is a symmetric positive definite matrix and Q is orthogonal. This is called the *polar decomposition* of A. Hint: Consider A^tA, which is a symmetric positive definite matrix, and take its square root. What is the complex version of this decomposition?

3.3. Suppose that $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$ is a bilinear form. Define the *left kernel* V_0 of $\langle \cdot, \cdot \rangle$ by
$$V_0 = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\},$$
and define the *right kernel* W_0 similarly. Prove that $\langle \cdot, \cdot \rangle$ induces a non-degenerate bilinear form on $V/V_0 \times W/W_0$, and in particular, $\dim(V/V_0) = \dim(W/W_0)$.

3.4. A *quadratic form* on a complex vector space V is a function $Q : V \to \mathbb{C}$ such that $Q(av) = a^2Q(v)$ for every $v \in V$ and $a \in \mathbb{C}$, and such that
$$\langle v, w \rangle := \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$$
is bilinear. If $\langle \cdot, \cdot \rangle$ is a bilinear form on V, prove that $Q(v) := \langle v, v \rangle$ is a quadratic form, and that every quadratic form on V arises in this way. Do Q and $\langle \cdot, \cdot \rangle$ uniquely determine each other?

3.5. The following statement is **false**: If $T : V \to V$ is an endomorphism, where V is finite dimensional over $\mathbb{C}$, then there exists a positive definite Hermitian inner product on V with respect to which T is normal. Explain why this statement is false, and add a hypothesis which makes the conclusion true. What happens if "normal" is changed to "unitary" or "Hermitian"?

3.6. Let P_d denote the vector space of polynomials of degree $\leq d$ over $\mathbb{R}$. Compute the signature of the symmetric bilinear form $\langle f, g\rangle = (fg)^{(d)}(0)$, where $(fg)^{(d)}(0)$ is the dth derivative of the product of polynomials f and g, evaluated at 0.

3.7. Let P_2 be the space of real polynomials of degree ≤ 2, equipped with the inner product $\langle f, g\rangle = \int_0^1 fg$. Find an orthonormal basis for P_2.

3.8. Prove that the set of $n \times n$ unitary matrices is a compact subset of $\mathbb{C}^{n^2}$.