Topology Lectures

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Abstract

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1 Introduction to topology

1.1 Topology of $\mathbb{R}$

We want to study continuity. We have the definition of a continuous function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, which is that for any $x_0 \in D$,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

We also have the notion of open and closed sets. An open set is a set U such that for any $x \in U$ there is a ball centered at x contained in U. A closed set is a set which contains all of its limit points, i.e. if $x_i \in F$ then $\lim_{i \to \infty} x_i \in F$. We recognize the following facts about open and closed sets and continuous functions f:

- Every open subset of $\mathbb{R}$ is the complement of a closed set and every closed subset is the complement of an open set. That is because if U is open and F is closed, we have for any $x_i \in U^C$, if $\lim x_i \in U$ then a ball around $\lim x_i$ is in U and hence some x_i is in U, a contradiction. Similarly, if $x \in F^C$ and every ball centered at x contained a point x in F, then there would be a sequence x_i whose limit is x and all of whose elements are in F. Thus $x = \lim x_i$ is in F since F is closed, which is a contradiction.
- If F is closed, then $f^{-1}(F)$ is closed. This is because if $x_i \in f^{-1}(F)$, and $x_\infty = \lim_{i \to \infty} x_i$ then $\lim_{i \to \infty} f(x_i) = f(x_\infty)$. Furthermore, $f(x_i) \in F$, so $f(x_0) \in F$. Thus $x_\infty \in f^{-1}(F)$.
- If U is open, then $f^{-1}(U)$ is open. This follows since $f^{-1}\left(U^C\right)$ is closed

and thus $\left[f^{-1}\left(U^C\right)\right]^C$ is open. Now, if

$$\begin{aligned} & x \in f^{-1}\left(U^C\right)^C \\ & \quad \Leftrightarrow x \notin f^{-1}\left(U^C\right) \\ & \quad \Leftrightarrow f(x) \notin U^C \\ & \quad \Leftrightarrow f(x) \in U \end{aligned}$$

which implies that $x \in f^{-1}(U)$, or $\left[f^{-1}\left(U^C\right)\right]^C = f^{-1}(U)$ is open.

- Every open set is the union of intervals. Arbitrary unions of open sets is open. This follows because every open set is the union of balls.

1.2 Definition of a topology

We use what we learned about the topology of $\mathbb{R}$ to define the general notions of a topology and of a continuous function. We define a topological space by specifying which sets are open. In order for this to make good sense, we must put a few conditions on which sets we may choose to be open.

Definition 1 *A topological space $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X which satisfy:*

1. *$X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$,*
2. *Arbitrary unions of sets $U \in \mathcal{T}$ are in $\mathcal{T}$, i.e. for any indexing set I, if $U_i \in \mathcal{T}$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \mathcal{T}$,*
3. *If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.*

Instead of explicitly writing $U \in \mathcal{T}$, we usually say that U is open. The complements of open sets are called closed.

Definition 2 *A set F is closed if $F^C = X \setminus F \in \mathcal{T}$, i.e. if F^C is open.*

Proposition 3 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 4 *A point $x \in X$ is a limit point of a set $A \subset X$ if every open set U containing x also contains a point $y \in A \setminus \{x\}$.*

Proposition 5 *$\bar{A}$ is equal to the union of A and its limit points.*

Proof. Let F be a closed set containing A. Then $X \setminus F$ is an open set disjoint from A, so if x is a limit point of A it cannot be in $X \setminus A$, thus all limit points are contained in $\bar{A}$ (which is the intersection of all closed sets containing A. Conversely, if $x \in \bar{A} \setminus A$ then if there were an open set U containing x but disjoint from A, then $\bar{A} \cap U^C$ is closed set strictly contained in $\bar{A}$ containing A, a contradiction since $\bar{A}$ is the smallest such set. ■

1.3 Construction of topologies

Let X be a topological space and $Y \subset X$ be a subset. We can give Y the *subspace topology* by saying a set $U \subset Y$ is open if $U = V \cap Y$ for some open set $V \subset X$. It is easy to show that this gives a topology. Think about how this gives a topology on the sphere $S^n \subset \mathbb{R}^{n+1}$.

Let X be a topological space and let $\sim$ be an equivalence relation. Recall that an equivalence relation $\sim$ is a relation satisfying the following properties:

1. (reflexivity) $x \sim x$.
2. (symmetry) $x \sim y$ implies $y \sim x$
3. (transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$.

Then $Q = X/\sim$ denotes the set of equivalence classes of the relation. There is a natural quotient map $q : X \to Q$ given by $q(x) = [x]$. The *quotient topology* is given by letting open sets be the sets $U \subset Q$ such that $q^{-1}(U) \subset X$ is open. One can specify the topology of the circle by considering $X = [0,1]$ and the equivalence relation $0 \sim 1$.

Another way to specify a topology is with a local base (system of neighborhoods), a basis or a subbasis.

Definition 6 *Let X be a set, and for every $x \in X$, let there be given a collection $\mathcal{N}(x)$ of subsets of X satisfying*

1. *$V \in \mathcal{N}(x) \implies x \in V$.*
2. *If $V_1, V_2 \in \mathcal{N}(x)$, then $\exists V_3 \in \mathcal{N}(x)$ such that $V_3 \subseteq V_1 \cap V_2$.*
3. *If $V \in \mathcal{N}(x)$, then there exists a $W \in \mathcal{N}(x)$ such that $W \subset V$ and if $y \in W$, then there exists $U \in \mathcal{N}(y)$ such that $U \subset V$.*

The collection $\{\mathcal{N}(x) | x \in X\}$ is a local base*.*

Given a local base, we can define a topology $\mathcal{T}$ by $O \in \mathcal{T}$ iff for all $x \in O$, there exists $V_x \in \mathcal{N}(x)$ such that $x \in V_x \subseteq O$.

Note that the neighborhoods of x in $\mathcal{N}(x)$ *do not have to be open*! However, given any local base, by "shrinking" the neighbourhoods a little if necessary, we can obtain a local base which generates the same topology, all of whose elements are open sets. In this case, condition 3 above simplifies to

$3'$. *If $V \in \mathcal{N}(x)$ and $y \in V$, then there exists $U \in \mathcal{N}(y)$ such that $U \subset V$.*

Definition 7 *A basis B is a collection of subsets of X such that for all $x \in X$, (1) there exists $U \in B$ such that $x \in U$ and (2) if $U, U' \in B$ and $x \in U \cap U'$, then there is another set $U'' \in B$ such that $x \in U''$ and $U'' \subset U \cap U'$. A basis generates a topology which can be described either as the union of all elements of B or such that a set $V \subset X$ is open if every point $x \in V$ has has a set $U \in B$ such that $x \in B \subset V$.*

An example of a basis is the open intervals for $\mathbb{R}$. Note that according to the first definition, the basis specifies the topology. Note that the sets in the basis have to be open, but the basis itself need not be a topology since unions of elements of the basis are not necessarily in the basis.

We can also specify a topology through a subbasis.

Definition 8 *A subbasis B' for a topology on X is a collection of sets whose union is X. The topology generated by the subbasis is such that every open set is a union of finite intersections of elements of B'.*

For a subbasis, the collection of finite intersections is a basis.

Let X and Y be topological spaces. We can give $X \times Y$ a topology by taking as a basis the sets $U \times V$ where $U \subset X$ and $V \subset Y$ are open sets. Note that not all open sets can be written as $U \times V$ for some $U \subset X$ and $V \subset Y$. This construction is the *product topology* (for finite cartesian products) and allows us to give a topology on $\mathbb{R}^n$ from the topology on $\mathbb{R}$.

Recall that a function $d : X \times X \to \mathbb{R}$ is a metric if for all $x, y, z \in X$, it satisfies:

1. (positive definite) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$
2. (symmetric) $d(x, y) = d(y, x)$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Recall that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$. The topology generated by the basis consisting of all balls $B(x, r)$ for all $r \in (0, \infty)$ and $x \in X$ is called the *metric topology*. (Show this is a basis.)

1.4 Continuous maps

We know that we can consider a continuous map from $\mathbb{R}$ to $\mathbb{R}$ to be one which whose graph does not require one to lift a pencil. It is easy to see that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map, then $f^{-1}(U)$ is open if U is open. Continuous maps between topological spaces are defined to be maps such that the inverse image of an open set is open.

Note that this implies that the inverse image of a closed set is closed (why?)

Continuous maps allow us to give an equivalence of topological spaces. We say that two topological spaces are homeomorphic if there exists a continuous map between them with a continuous inverse. Such a map is called a homeomorphism. Show that the circle given by the subspace topology from $\mathbb{R}^2$ is homeomorphic to the quotient topology $[0, 1] / \sim$ as above.

1.5 Induced topologies

Proposition 9 *If X is a set, $(Y, \mathcal{S})$ is a topological space, and $p : Y \to X$ is a function, then we can define a topology $\mathcal{T}$ on X by $U \subseteq X$ is open iff $p^{-1}(U) \in \mathcal{S}$.*

With this construction p is a continuous function.

We can also go in the other direction. If $f_\alpha : X \to (Y_\alpha, \mathcal{S}_\alpha$ is a indexed family of functions, then we can define the induced (or *weak*) topology on X induced by the family of functions by defining a subbasis B' by

$$B' = \{f_\alpha^{-1}(U_\alpha) | U_\alpha \in \mathcal{S}_\alpha\}$$

Verify that this is indeed a subbasis.

Proposition 10 *The* weak *topology constructed above is the coarsest topology on X that makes all the functions f_α continuous.*

If $Z \subseteq X$, and $(X, \mathcal{T})$ is a topological space, then the inclusion map $i : Z \to X$ induces the *relative* or *subspace* topology on Z. Give a characterization of the open sets in this topology.

1.6 Separation and countability

Here we simply list some of the separation and countability properties.

Separation:

- Hausdorff. A space is Hausdorff if for every two points $x, y \in X$, there are disjoint open sets U and V such that $x \in U$, $y \in V$, $x \notin V$, $y \notin U$. Note that a subspace of a Hausdorff space is Hausdorff but the quotient of a Hausdorff space may not be Hausdorff (example: line with two origins).

- Regular. A space is regular if one point sets are closed and for each pair of a point x and a closed set B disjoint from x there are disjoint open sets containing x and B.

- Normal. A space is normal if one point sets are closed and for each pair of disjoint closed sets A, B there are disjoint open sets containing A and B.

Hausdorff is the most important. One reason is the following.

Proposition 11 *Finite point sets in Hausdorff spaces are closed.*

We note some examples of Hausdorff, regular, and normal spaces.

Countability. A set is countable if there is a bijection between it and the natural numbers. It is easy to see that the integers, the even integers, and the rational numbers are all countable sets. It is also possible to see that the real numbers between 0 and 1 form an uncountable set using Cantor's diagonal argument. Topological spaces have the following countability axioms:

- First countable. A space is first countable if every point has a countable basis, i.e. given $x \in X$ there is a countable collection of open sets $U_1, U_2, U_3, \ldots$ such that for any neighborhood V of x, there is $k \in \mathbb{N}$ such that $U_k \in V$.

- Second countable. A space is second countable if it has a countable basis for the topology. (Long line is an example which is not second countable.)

1.7 More exotic examples

- Discrete topology. All points are open sets.
- Indiscrete topology. The only open sets are X and $\varnothing$.
- Line with two origins. We consider two real lines, $\mathbb{R} = \{x : x \in \mathbb{R}\}$ and $\mathbb{R}' = \{x' : x \in \mathbb{R}\}$. The line with two origins is the quotient $\mathbb{R} \cup \mathbb{R}' / \sim$ where $x \sim x'$ if $x \neq 0$. Show that this space is not Hausdorff.
- Order topology. A total ordering on a set X is a relation $\leq$ such that for any $x, x' \in X$ we have either that $x \leq x'$ or $x' \leq x$ and both are true if and only if $x = x'$, and the relation is transitive. The order topology is the topology generated by the basis of intervals $(a, b) = \{x \in X : a \leq x \text{ and } x \leq b\}$. Products of ordered sets can be given the dictionary order. What do you think the definition of the dictionary order is? Is the order topology on $\mathbb{R}^2$ Hausdorff?
- Finite complement topology. The finite complement topology is the topology such that the complement of finite sets are open. We can also consider the countable complement topology.
- Zariski topology. Consider the following topology on $\mathbb{R}^n$. We take as the closed sets the sets

  $$F(S) = \{x \in \mathbb{R}^n : f(x) = 0 \forall f \in S\}$$

  where S is a set of polynomials in n variables. Show that this is a topology on $\mathbb{R}^n$. Show that any two open sets must intersect, and hence the topology cannot be Hausdorff.

1.8 The Vocabulary of topology

$(X, \mathcal{T})$ is a topological space. If $O \in \mathcal{T}$, then O is said to be *Open.* If $F \subseteq X$ is such that F^c is open, then F is said to be *closed*.

If $A \subseteq X$, the union of all the open sets contained in A is called the interior of A, and is denoted by A°. Likewise, the intersection of all the closed sets containing A is called the closure of A and is denoted by $\bar{A}$ or by $\mathrm{cl}(A)$.

Proposition 12 *The interior of A is open and the closure of A is closed. They are, respectively, the largest open set contained in A (smallest closed set containing A).*

A sequence $x_n \in X$ converges to a point $y \in X$ if for all open sets $O \ni y$, there exists an index $M < \infty$ such that for all $n > M$, $x_n \in O$. It is enough to check this for every neighbourhood of y, so that, if $\mathcal{N}(y)$ is a local base for the topology, it suffices to check that for all $U \in \mathcal{N}(y)$, there exists an index $P < \infty$ such that for all $n > P$, $x_n \in U$. This is a direct generalization of the notion of convergence in metric spaces.

There however, is no generalization of the notion of a Cauchy sequence to a topological space, since this requires the ability to compare the "size" of neighbourhoods at distinct points, and a Topological structure does not allow for this comparison ("Rubber sheet geometry"). For example, show that, $(0,1)$ is homeomorphic to $\mathbb{R}$ but there exist Cauchy sequences in $(0,1)$ (with the usual metric), whose images in $\mathbb{R}$ are not Cauchy.

$x \in A \subseteq X$ is an *isolated point* if there is an open sect O such that $O \cap A = \{x\}$. $x \in X$ is an *accumulation point* of A if there exists a sequence in $A/\{x\}$ that converges to x.

2 Compactness

2.1 Closed and bounded sets in $\mathbb{R}^n$

We notice that closed and bounded subsets X of $\mathbb{R}^n$ have the following very nice properties:

- Every sequence contained in X has a limit point in X. That is, every sequence has a subsequence which converges to a point in X.
- The set X can be covered by finitely many open sets. In fact, we can do better. Given any open cover, there is a finite subset of that cover which also covers (finite subcover).
- Every continuous function on X attains a maximum and a minimum on X.

We shall use the middle definition to define compactness on a general topology.

2.2 General definitions of compactness

Definition 13 *A topological space X is* compact *if any open cover of X has a finite subcover, i.e. for any collection of open sets $\{U_i\}_{i\in I}$ with $X \subset \bigcup_{i\in I} U_i$, there exists a finite subcollection $\{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\} \subset \{U_i\}_{i\in I}$ such that $X \subset \bigcup_{j=1}^{k} U_{i_j}$.*

This may seem a rather abstract definition but it has many important properties which follow immediately. Let us take for granted for the moment that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. As a warm-up, we have the following.

Proposition 14 *A compact metric space can be covered by finitely many balls of any given radius.*

Proof. For any compact metric space X and $r > 0$, $\bigcup_{x\in X} B(x,r)$ is and open cover of X. Thus it has a finite subcover and so there exist $\{x_i\}_{i=1}^{k}$ such that $X = B(x_1,r) \cup \cdots \cup B(x_k,r)$. ■

2.3 Properties of compact spaces

In this section we see some properties of a compact set.

Proposition 15 *If X is compact and $F \subset X$ is closed, then F is compact.*
Proof. *Given an open cover $\{U_i\}_{i\in I}$ of F, there is a cover $\left\{\tilde{U}_i\right\}_{i\in F}$ such that $\tilde{U}_i$ are open in X and $U_i = \tilde{U}_i \cap F$. Consider $\left\{\tilde{U}_i\right\}_{i\in F} \cup (X \setminus F)$. This is an open cover and so it must contain a finite subcover. Since F is not contained in $X \setminus F$, it must be contained in the finite subcover $\left\{\tilde{U}_{i_j}\right\}_{j=1}^{k}$ and hence $\left\{U_{i_j}\right\}_{j=1}^{k}$.*
■

Proposition 16 *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact (with the subspace topology).*
Proof. *If $\{U_i\}_{i\in I}$ is an open cover of $f(X)$, then $U_i = \tilde{U}_i \cap f(X)$ where $\tilde{U}_i$ are open in Y. We see that $\left\{f^{-1}\left(\tilde{U}_i\right)\right\}_{i\in I}$ is an open cover of X, thus it must have a finite subcover $\left\{f^{-1}\left(\tilde{U}_{i_j}\right)\right\}_{j=1}^{k}$. But then $\left\{\tilde{U}_{i_j}\right\}_{j=1}^{k}$ must cover $f(X)$ (given $y = f(x)$, $x \in f^{-1}\left(\tilde{U}_{i_j}\right)$ and thus $y = f(x) \in \tilde{U}_{i_j}$) and thus $\left\{U_{i_j}\right\}_{j=1}^{k}$ is a finite subcover of $f(X)$.* ■

Proposition 17 *If $f : X \to \mathbb{R}$ is continuous and X is compact, then there exist x_m and x_M in X such that*

$$f(x_m) = \inf_{x\in X} f(x)$$
$$f(x_M) = \sup_{x\in X} f(x).$$

Proof. *Since $f(X)$ is compact, it must be closed and bounded and thus it contains its lower and upper bounds.* ■

Proposition 18 *If X and Y are compact, then $X \times Y$ is compact.*

Proof. We know that $\{x\} \times Y$ is homeomorphic to Y and is thus compact. Given an open cover C of $X \times Y$, we know that it must also cover $\{x\} \times Y$. Since $\{x\} \times Y$ is compact, there is a finite subcover $\{C_1, \ldots, C_k\}$ which cover $\{x\} \times Y$. We will need to show the following: Given any open set N containing $\{x\} \times Y$, there is an open set $W \subset X$ such that $\{x\} \times Y \subset W \times Y \subset N$. Suppose we have this fact, then we take $N = C_1 \cup \cdots \cup C_k$, and so there exists W as stated. Hence $W \times Y$ is covered by $\{C_1, \ldots, C_k\}$. For each x there is a W_x such that $\{x\} \times Y \subset W_x \times Y$ and $W_x \times Y$ is covered by finitely many sets. the sets $\{W_x\}_{x \in X}$ form a cover of X, and hence there is a finite subcover $\{W_1, \ldots, W_\ell\}$. The sets $\{W_1 \times Y, \ldots, W_\ell \times Y\}$ cover $X \times Y$ and each can be covered by finitely many sets in C, so the proposition is proved provided we prove the missing statement.

Suppose N is an open set containing $\{x\} \times Y$. By the definition of the product topology on $X \times Y$, around every point in $\{x\} \times Y$ there is a basis element $U \times V$ which is contained in N. Since $\{x\} \times Y$ is compact, we can take finitely many basis elements so that $\{U_1 \times V_1, \ldots, U_j \times V_j\}$ cover $\{x\} \times Y$. We may assume that each $U_i \times V_i$ intersects $\{x\} \times Y$ since otherwise it may be discarded. Now let $W = U_1 \cap \cdots \cap U_j$. Clearly this set is open and contains x_0. Finally we must show that $\{U_1 \times V_1, \ldots, U_j \times V_j\}$ actually covers $W \times Y$. Given $(w, y) \in W \times Y$, we have that $(x_0, y) \in U_i \times V_i$ for some i, but then also $w \in U_i$ (since w is in every U_i) and so $(w, y) \in U_i \times Y_i$ and the proof is done. ■

Proposition 19 *Compact subsets of a Hausdorff space are closed.*

2.4 Heine-Borel theorem

Notice that $(0,1]$ is not compact, because the cover $\bigcup_{k \in \mathbb{N}} \{(1/k, 1]\}$ has no finite subcovers.

Theorem 20 *(Heine-Borel)* $[0,1]$ *is compact.*

Proof. Let $\mathcal{U}$ be a cover of $[0,1]$. We let

$$S = \{x \in [0,1] : [0,x] \text{ has a finite subcover in } \mathcal{U}\}.$$

Now we show that $y = \sup S$ must be 1. Observe since $\mathcal{U}$ is a cover, y is contained in some open set U, and hence the interval $(y - \varepsilon, y + \varepsilon) \subset U$ for some small $\varepsilon > 0$. This implies both that $y \in S$ since there must be some $y' \in S$, $y' > y - \varepsilon$ since y is the sup, so take the finite cover of $[0, y']$ and add in U. But this also implies that $y + \varepsilon/2 \in S$ if $y + \varepsilon/2 \in [0,1]$, so $y = 1$. ■

Definition 21 *A subset X of $\mathbb{R}^n$ is said to be* bounded *if there exists $r > 0$ such that $X \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.*

Corollary 22 *Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.*

Proof. Closed and bounded sets are closed subsets of some compact set $[-k,k]^n$, and thus compact. Conversely, if a subset of $\mathbb{R}^n$ is compact, it must be closed since $\mathbb{R}^n$ is Hausdorff and it must be bounded because we can take the cover of $(-k,k)^n$ for all $k = 1, 2, \ldots$ and it must have a finite subcover. ■

2.5 Sequences, continuity and compactness

Definition 23 *A function* $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ *is* sequentially continuous *if for every convergent sequence* $x_n \to x$ *in* X*, we have* $f(x_n) \to f(x)$*.*

Proposition 24 *Every continuous function is sequentially continuous. In a first countable space (for example, in a metric space), the converse is also true.*

Definition 25 *A space* X *is* sequentially compact *if every sequence in* X *has a convergent subsequence.*

Proposition 26 *Every compact space is necessarily sequentially compact. In a metric space, the converse is also true.*

Taken in conjunction with the Heine-Borel theorem, this implies that

Theorem 27 *(Bolzano-Weierstrass) Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

2.6 Examples of compact spaces

- Any finite topological space is compact.
- The finite-dimensional sphere $\left\{x \in \mathbb{R}^n : |x|^2 = 1\right\}$.
- The Cantor set is compact.
- The finite-complement topology is compact.

3 Connectedness

3.1 Connected and disconnected sets in $\mathbb{R}^n$

The key property of a connected set is the intermediate value theorem, which states that if $f : [a, b] \to \mathbb{R}$ is a continuous function and $f(a) \leq r \leq f(b)$ then there exists $c \in [a, b]$ such that $f(c) = r$. Notice this is not true for functions on disconnected sets such as $(0, 1) \cup (1, 2)$.

3.2 Definition of connected

Definition 28 *A separation of a space* X *is a pair* U, V *of disjoint open subsets of* X *such that* $X = U \cup V$*. Note that the two sets* U *and* V *are both open and closed since* $U = X \setminus V$ *and* $V = X \setminus U$*. The trivial separation consists of* X *and* $\varnothing$*.*

Definition 29 *A space* X *is* connected *if there exist no nontrivial separations of* X*. Equivalently,* X *is connected if the only open and closed subsets of* X *are* X *and* $\varnothing$ *(since if* $A \subset X$ *is open and closed, then* $X = A \cup A^C$ *is a separation if neither is empty). A space which is not connected is said to be* disconnected.

Example 30 $(0,1)$ *is connected.*

Example 31 $(0,2)\backslash\{1\}$ *is disconnected since* $(0,1)$ *and* $(1,2)$ *form a nontrivial separation.*

A note on the proof of Heine-Borel: we essentially used that $[0,1]$ is connected and showed that the set of points $y \in [0,1]$ such that $[0,y]$ can be covered by a finite subcover is both open and closed, and hence must be everything.

3.3 Properties of connected sets

Proposition 32 *The union of connected sets which each have a common point is connected.*

Proof. Let $X = \bigcup X_i$ such that $x \in \bigcap X_i$ and X_i are connected. Now suppose that $X = U \cup V$ where U and V are disjoint open sets. Since U and V are disjoint, $x \in U$ or V. Say $x \in U$. Then $U \cap X_i$ and $V \cap X_i$ are a disjoint open cover of X_i, and so $X_i \subset U$ (since $x \in U$). This is true for all i, so $X \subset U$. ■

Proposition 33 *Let* A *be a connected subset of* X. *If* $A \subset B \subset \bar{A}$ *then* B *is connected.*

Proof. Suppose $B = U \cup V$, where U and V are disjoint and open. Then $U \cap A$ and $U \cap B$ are disjoint and open and cover A, so $A \subset U$ or $A \subset V$, say $A \subset U$. So $\bar{A} \subset \bar{U} = U$, since $U = X \setminus V$ is closed. ■

Proposition 34 *The product of connected sets is connected.*

Proof. We see that $\{x\} \times Y$ and $X \times \{y\}$ are connected. Since both contain (x,y), $V_{x,y} = (\{x\} \times Y) \cup (X \times \{y\})$ is connected. Now we see that

$$\bigcup_{x \in X} V_{x,y} = X \times Y$$

and

$$\bigcap_{x \in X} V_{x,y} = X \times \{y\} \neq \varnothing.$$

Thus $X \times Y$ is connected. ■

Proposition 35 *If* $f : X \to Y$ *is continuous and* X *is connected then* $f(X)$ *is connected.*

Proof. If $A \subset f(X)$ is nonempty and both open and closed, then since f is continuous, $f^{-1}(A)$ is both open and closed, and hence $f^{-1}(A) = X$. But then $A = f(X)$. Thus $f(X)$ is connected. ■

Proposition 36 *(Intermediate value theorem) If* X *is connected,* $f : X \to \mathbb{R}$ *is continuous, and* $f(a) \leq r \leq f(b)$ *then there exists* $c \in X$ *such that* $f(c) = r$.

Proof. We know that $f(X)$ is a connected subset of $\mathbb{R}$. Now if there is no c such that $f(c) = r$, then we can cover $f(X)$ by the sets $(-\infty, r) \cap f(X)$ and $(r, \infty) \cap f(X)$, which are disjoint open sets. They are nonempty since one contains $f(a)$ and the other $f(b)$. This is a separation, contradicting that $f(X)$ is connected. ■

3.4 Path connected

Definition 37 *A path in X is a continuous map $\gamma : [a, b] \to X$.*

Definition 38 *A space X is* path connected *if any two points can be joined by a path.*

We shall see that path connected is weaker than connected, but first we show that path connected implies connected.

Proposition 39 *If X is path connected, then it is connected.*

Proof. We shall show that if X is not connected, then it is not path connected. If X is not connected, then there is a separation $\{U, V\}$. Given $x, y \in X$ if there were a path $\gamma : [a, b] \to X$ between them, then $\gamma([a, b])$ would be connected, which implies the path must lie entirely in U or V (otherwise U and V would form a separation for $\gamma([a, b])$), which says that there are no paths between points in U and points in V. Hence X is not path connected. ■

3.5 Components

Definition 40 *Given X, we can define an equivalence relation on X by setting $x \sim y$ if there is a connected subset containing both x and y. The equivalence classes are called* components *or* connected components *of X.*

Show that this is an equivalence relation.

Proposition 41 *The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one component.*

Proof. Let $\{C_i\}_{i \in I}$ be the components. Since the components are equivalence classes, they must be disjoint and must cover. If U is connected and $x_i \in U \cap C_i$ and $x_j \in U \cap C_j$ then $x_i \sim x_j$, which implies that $C_i = C_j$ by the definition of components. Now we must show that components are connected. Fix $x_0 \in C_i$. For any $x \in C_i$, there is a connected set A_x containing both x_0 and x since $x \sim x_0$. Thus $C_i = \bigcup_{x \sim x_0} A_x$, which implies that C_i is connected since it is the union of connected sets with a common intersection point x_0. ■

We can also look at path components.

Definition 42 *Define an equivalence relation on X by $x \sim y$ if there is a path from x to y. The equivalence classes are called* path components *of X.*