Topology problem set – Integration workshop 2009

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1 Topological spaces and Continuous functions

1.1 If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, show that $(X, \mathcal{T}_1 \cap \mathcal{T}_2)$ is also a topological space. Give an example where $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X.

1.2 The structure of open sets in $\mathbb{R}$.

In what follows, we are considering the standard (metric) topology on $\mathbb{R}$.

(a) Let S be a nonempty open subset of $\mathbb{R}$. For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Show that, A_x and B_x are both non-empty.

(b) Where $x \in S$ as above, if A_x is bounded below, let $a_x = \inf(A_x)$. Otherwise, let $a_x = -\infty$, and define b_x is a corresponding manner. Show that $x \in I_x = (a_x, b_x) \subseteq S$.

(c) Show that $S = \cup_x I_x$.

(d) Show that the intervals I_x give a partition of S, *i.e.*, for $x, y \in S$, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

(e) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.

(f) Prove that every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

1.3 $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are two topological spaces.

(a) If $W \subseteq X$ is a subset, show that the *relative* or *subspace* topology of W defined by $O \subseteq W$ is open only if $O = W \bigcap U$ for some $U \in \mathcal{T}$ is indeed a topology.

(b) If (X, d) is a metric space, and $W \subseteq X$, then the restriction of d to W gives a metric space. We can define two topologies on W, the relative topology on W from the metric topology on X, and the metric topology on W induced by the restriction of d onto W. Show that the two topologies are the same.

(c) For every $(x, y) \in X \times Y$, define the collection of neighborhoods by

$$\mathcal{N}(x, y) = \{U_x \times V_y | x \in U_x \in \mathcal{T}, y \in V_y \in \mathcal{S}\}$$

Show that this collection of neighborhoods defines a *local base*. The topology generated by this local base is called the *product topology*.

1.4 Product topology Let X be a collection of topological spaces indexed by a set $\mathcal{I}$. We define a topology on the Cartesian product $\prod_{\alpha \in \mathcal{I}} X$ as follows: a basis is given by sets of the form $\prod_{\alpha \in \mathcal{I}} U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open and for all but finitely many α , $U_\alpha = X$.

(a) Prove that the above construction does indeed yield a basis.

(b) Prove that this is the weakest topology (with fewest open sets) such that the projections $\pi_\alpha : \prod_{\alpha\in\mathcal{I}} X_\alpha \to X_\alpha$ are continuous.

1.5 Comparing topologies $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are topological spaces. The topology $\mathcal{T}_1$ is said to be *finer* than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

(a) Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if for every $x \in X$ and $(U \ni x) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ such that $V \subseteq U$.

(b) Show that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ if and only if the identity map $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

(c) Show that the l^1 and l^2 metrics on $\mathbb{R}^2$ generate the same topology.

1.6 If $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ and $g : (X, \mathcal{S}) \to (Z, \mathcal{V})$ are continuous, show that the composition $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{V})$ is also continuous.

1.7 A set in a topological space is *closed* if it's complement is open. If $f : X \to \mathbb{R}$ is a continuous function, show that $f^{-1}([0, 1])$ is closed in X.

1.8 Local base for a topological space

(a) $\mathcal{N}(x)$ is a local base for a topological space (X, T). If $A \in \mathcal{N}(x)$, show that there exists a $U \in T$ such that $x \in U \subseteq A$.

(b) $\mathcal{N}_1(x)$ and $\mathcal{N}_2(x)$ are local bases for a space X. Show that the topology T_1 generated by $\mathcal{N}_1(x)$ is finer that the topology T_2 generated by $\mathcal{N}_2(x)$ if and only if for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ such that $x \in A \subseteq B$.

1.9 First Countable local base

$(X, \mathcal{T})$ is first countable, if it has a local base $\mathcal{N}(x)$ such that at every point $x \in X$, the collection of neighborhoods $\mathcal{N}(x)$ is countable.

(a) Show that the metric topology on a metric space (X, d) is first countable.

(b) Is $(X, \mathcal{T})$ is first countable, show that every point $x \in X$ has a countable collection of open neighborhoods $U_n \ni x$ such that $U_n \subseteq U_{n+1}$, and x is an interior point for an open set O if and only if there is an index n such that $x \in U_n \subseteq O$.

(c) With the same defintitions as the previous part, show that if you construct a sequence by picking arbitrary points $y_n \in U_n$, it follows that the sequence $\{y_n\}$ converges.

(d) If $(X, \mathcal{T})$ is first countable, and $(Y, \mathcal{S})$ is any topological space, show that $f : X \to Y$ is continuous if and only if it is sequentially continuous.

1.10 Second countable spaces Recall that a topological space is *second countable* if it has a countable base.

(a) Show that $\mathbb{R}$ with the usual topology is second countable.

(b) Give an example of a space which is first countable but not second countable.

(c) Show that every second countable space is also first countable.

(d) A subspace A of X is called dense if the closure of A is X. A topological space X is called *separable*, if there exists a countable dense subset. Show that if X is second countable, then X is separable.

1.11 A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semi continuous* if for all $x \in \mathbb{R}, \epsilon > 0$, there exists a $\delta > 0$ such that $|y - x| < \delta$ implies that $f(y) > f(x) - \epsilon$.

(a) Show that the collection $\mathcal{B} = \{(\alpha, \infty) \,|\, \alpha \in \mathbb{R}\}$ is a base. Let T' denote the topology generated by $\mathcal{B}$. Show that $T' \subset T_{metric}$ and the containment is strict. (Hint: One idea is to show that T' is not Hausdorff.)

(b) Show that the collection $\mathcal{B}$ along with the empty set and all of $\mathbb{R}$ is the topology generated by the base $\mathcal{B}$, *i.e.* $T' = \mathcal{B} \cup \{\emptyset, \mathbb{R}\}$.

(c) Show that T' is second countable.

(d) Show that a function $f : (\mathbb{R}, T_{metric}) \to (\mathbb{R}, T')$ is continuous, if and only if it is lower semi-continuous by the earlier definition.

(e) Show that a function $f : (\mathbb{R}, T') \to (\mathbb{R}, T_{metric})$ is continuous, if and only if it is a constant function.

1.12 **Accumulation points** $A \subseteq \mathbb{R}$, and A' denotes the set of all the accumulation points of A.

(a) If $y \in A'$ and $U \subseteq \mathbb{R}$ is an open set containing y, show that there are infinitely many distinct points in $A \cap U$.

(b) Show that

$$A' = \bigcap_{x \in A} cl(A \backslash \{x\}).$$

(c) Using this, or otherwise, show that A' is a closed set.

(d) Show that $cl(A) = A \cup A'$.

1.13 **A topology on** $\mathbb{N}$

Let $X = \mathbb{N} \cup \{e\}$. Define a collection $\mathcal{T}$ by $A \subseteq X$ is in T if and only if A does not contain e (this includes the empty set) or $e \in A$ and A^c is finite (this includes X).

(a) Show that T is a topology on X.

(b) Show that T is second countable.

(c) Show that $\mathbb{N}$ is dense in (X, T).

(d) A function $f : \mathbb{N} \to \mathbb{R}$ is the same thing as a sequence x_n. We will say that $g : X \to \mathbb{R}$ is a continuous extension of f if $g(n) = f(n)\, \forall n \in \mathbb{N}$. Show that f has a continuous extension iff $x_n = f(n)$ is a convergent sequence. Further, the continuous extension is given by $g(e) = \lim_{n \to \infty} f(n)$.

(e) Every element $a = (l, a_1, a_2, a_3, \ldots) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ defines a function $f_a : X \to \mathbb{R}$ by $f_a(n) = a_n, f_a(e) = l$. Let $Y \subset \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ denote the set of all the convergent sequences with their associated limits, *i.e.* $(l, a_1, a_2, a_3, \ldots) \in Y \implies a_n \to l$. Find the weakest topology on X such that for all $a \in Y$, $f_a : X \to \mathbb{R}$ is continuous.

(f) Can you find a metric on X such that the metric topology is identical to the topology T above? Any topology with this property is said to be *metrizable*.

1.14 A topological space X is called a T1-space (or a Tychonoff space) if for any two different points $x, y \in X$ there exists an open set U which contains x but does not contain y. Prove that a space X is a T1-space if and only if any subset consisting of a single point is closed. Also find an example of a topological space which is a T1-space, but not Hausdorff (a T2-space).

1.15 Define the profinite topology on $\mathbb{Z}$ in which the open sets are the empty set and unions of arithmetic progressions.

(a) Show that an arithmetic progression is also a closed set in this topology.

(b) Show that if there were only finitely many primes, then the set $\{-1, 1\}$ would be open.

(c) Then show that this set is not open and conclude that there are infinitely many primes.

(d) Let T^∞ be the product of countably infinitely many copies of the unit circle with the product topology. Define the map $\phi : \mathbb{Z} \to T^\infty$ as follows:

$$\phi(n) = (\exp(2\pi i n/2), \exp(2\pi i n/3), \exp(2\pi i n/4), \exp(2\pi i n/5), ...) .$$

Show that this map is injective and the induced topology on $\mathbb{Z}$ coincides with the profinite topology.

1.16 **Zariski topology** Consider the topology on $\mathbb{R}^n$ in which the open sets are the empty set and the complements of the common zero levels sets of finitely many polynomials. Show that this is indeed a topology on R^n. This is called the Zariski topology. Show also that the Zariski topology is not Hausdorff.

1.17 Let the group $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$t.(x, y) = (x, y + tx).$$

Prove that the quotient space with the quotient topology is not Hausdorff, but is the union of two disjoint Hausdorff subspaces. Also show that the quotient space is a T1-space.

2 Compactness

2.1 Prove or disprove the following:

(a) A is finite and U is a open subset of $\mathbb{R}$. If $A \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in A$, $N(x, \epsilon) \subseteq U$.

(b) P is countable and U is a open subset of $\mathbb{R}$. If $P \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in P$, $N(x, \epsilon) \subseteq U$.

(c) F is closed and U is a open subset of $\mathbb{R}$. If $F \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in F$, $N(x, \epsilon) \subseteq U$.

(d) K is compact and U is a open subset of $\mathbb{R}$. If $K \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in K$, $N(x, \epsilon) \subseteq U$.

2.2 If X is compact and $f : X \to Y$ is continuous, show that $f(X)$ is compact.

2.3 Prove that the unit sphere in $\mathbb{R}^n$ is compact.

2.4 Consider the topology on X in which the open sets are the empty set and the complements of finite subsets. Show that every subset of X is compact, although not every subset of X is closed, in general.

2.5 Let $\mathbb{RP}^n$ denote the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$, by the equivalence relation $x \sim y$ iff $\exists \lambda \neq 0$, s.t. $x = \lambda y$. Show that $\mathbb{RP}^n$ is compact.

2.6 Show that if X is compact and Y is Hausdorff and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

2.7 Show that every compact subset of a Hausdorff space is closed.

2.8 A space is *locally compact* if every point has a compact neighbourhood.

(a) If X is a compact space, then show that X is locally compact.

(b) Give an example of a space which is not locally compact.

(c) There is a canonical way to add one point to a locally compact Hausdorff space to get a compact space. Namely, if X is locally compact Hausdorff, let $\bar{X} = X \cup \{\infty\}$. The open sets of $\bar{X}$ are the open sets of X together with the sets $(X \setminus K) \cup \{\infty\}$, where K is a compact subset of X. Prove that $\bar{X}$, called the *one point compactification* of X, is a compact Hausdorff space.

2.9 A metric space X is said to be *totally bounded* if for any $\epsilon > 0$ it can be covered by finitely many ϵ-balls. Also, a metric space is complete, if every Cauchy sequence in X converges. Prove that X is compact if and only if X is complete and totally bounded.

3 Connectedness

3.1 Show that the Cantor set is totally disconnected.

3.2 If X is connected and $f : X \to Y$ is continuous, show that $f(X)$ is connected.

3.3 Find all the different topologies, up to homeomorphism, on a 4-element set, which make it a connected topological space.

3.4 Prove that the closure of a connected subspace is connected.

3.5 Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Hint: use the notion of a connected set.

3.6 Prove that each connected component of a topological space X is closed.

3.7 Show that if A is a both open and closed, non-empty, connected subset of a topological space X, then A is a connected component.

3.8 Show that if a topological space has finitely many connected components, then each of them is open and closed.

3.9 A space X is called locally path-connected, if for each $x \in X$ and every neighbourhood U of x, there exists a path-connected neighbourhood V of x contained in U. Show that if X is connected and locally path-connected, then it is path-connected.

3.10 Show that if K is the Cantor set, then the complement of $K \times K$ in the unit square $[0, 1] \times [0, 1]$ is path-connected.