Polynomial Differential Forms for Efficient Finite Element Methods

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joint work with
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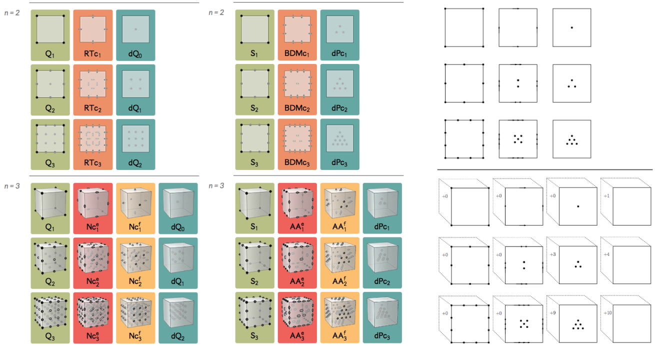


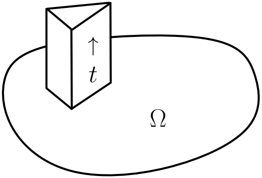
Table of Contents

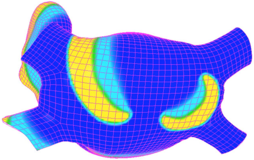
Outline

1. Polynomial spaces for finite element methods
2. The "Periodic Table of the Finite Elements"
3. Trimmed serendipity finite elements
4. Computational bases for serendipity-type spaces

What are (efficient) finite element methods?

The **finite element method** is a way to numerically approximate the solution to PDEs.





$$\begin{bmatrix} & \mathbb{A} & \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \end{bmatrix}$$

CHARACTERIZE	DISCRETIZE	SOLVE
Real analysis	Geometry & Topology	Linear algebra
PDEs	Combinatorics	Numerical analysis

Order of accuracy of computed solution $\rightarrow$ depends on local "basis" functions on each element.

Size of the linear system $\rightarrow$ depends on the number of mesh elements and the number of degrees of freedom associated to each element.

For computational efficiency: maximize order of accuracy while minimizing degrees of freedom.

The Finite Element Method: 1D

Ex: The 1D Laplace equation: find $u(x) \in U$ s.t.

$$\begin{cases} -u''(x) = f(x) & \text{on } [a,b] \\ u(a) = 0, \\ u(b) = 0 \end{cases}$$

Make the problem easier by making it (seemingly) harder . . .

Weak form: find $u(x) \in U$ ($\dim U = \infty$) s.t.

$$\int_a^b u'(x)v'(x)\,dx = \int_a^b f(x)v(x)\,dx, \quad \forall v \in V \quad (\dim V = \infty)$$

. . . but we can now search a finite-dimensional space. . .

Discrete form: find $u_h(x) \in U_h$ ($\dim U_h < \infty$) s.t.

$$\int_a^b u_h'(x)v_h'(x)\,dx = \int_a^b f(x)v_h(x)\,dx, \quad \forall v_h \in V_h \quad (\dim V_h < \infty)$$

Typical approach: $U_h = V_h =$ (some space of piecewise polynomials)

The Finite Element Method: 1D

Suppose $u_h(x)$ can be written as linear combination of V_h elements:

$$u_h(x) = \sum_{v_i \in V_h} u_i v_i(x)$$

The discrete form becomes: find coefficients $u_i \in \mathbb{R}$ such that

$$\sum_i \int_a^b u_i v_i'(x) v_j'(x)\, dx = \int_a^b f(x) v_j(x) \;\, dx, \quad \forall v_j \in \text{basis for } V_h \quad (\dim V_h < \infty)$$

Written as a linear system:

$$[\,\mathbb{K}\,]_{ji} \;\; [\,u\,]_i = [\,f\,]_j\,, \quad \forall v_j \in \text{basis for } V_h$$

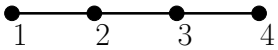
With some functional analysis we can prove: $\exists\; C > 0$, independent of h, s.t.

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\substack{\text{error between cnts} \\ \text{and discrete solution}}} \quad \leq \quad \underbrace{C\, h^{\,p}\, |u|_{H^2(\Omega)}}_{\substack{\text{bound in terms of} \\ \text{2nd order osc. of } u}} \quad , \qquad \underbrace{\forall u \in H^2(\Omega)}_{\substack{\text{holds for any } u \text{ with} \\ \text{bounded 2nd derivs.}}} \qquad .$$

where h = maximum width of elements use in discretization
and p depends on choice of space V_h

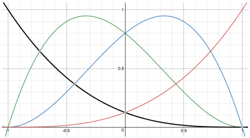
Choosing a finite element type: 1D

Set V_h $:=$ piecewise polynomials, max degree p on each segment, constrained to meet with C^0 continuity at vertices.

$\{1, x, x^2, x^3\}$	$\longleftrightarrow$ *change of basis*	$\{\psi_1, \psi_2, \psi_3, \psi_4\}$	$\longleftrightarrow$ *associate to geometry*	
monomials	$\longleftrightarrow$	**basis functions**	$\longleftrightarrow$	**domain points**

Cubic Basis on $[-1, 1]$

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} := \begin{bmatrix} (x-1)^3 \\ (x-1)^2(x+1) \\ (x-1)\,(x+1)^2 \\ (x+1)^3 \end{bmatrix} \rightarrow (\textit{scale}) \rightarrow$$

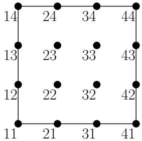


$\rightarrow$ Observe ϕ_1, ϕ_4 interpolate values at endpoints while ϕ_2, ϕ_3 are associated to "interior" approximation.

$\rightarrow$ Straightforward in 1D to generalize to arbitrary $p \geq 1$ or continuity C^1, C^2, etc.

Cubic order tensor product basis functions: 2D

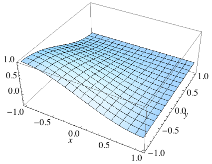
$$\underbrace{\left\{ \begin{array}{c} x^r y^s \\ 0 \leq r,s \leq 3 \end{array} \right\}}_{\mathcal{Q}_3([0,1]^2)} \longleftrightarrow \left\{ \begin{array}{c} \psi_{ij}(x,y) \\ 1 \leq i,j \leq 4 \end{array} \right\} \longleftrightarrow$$



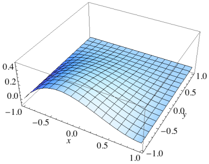
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: For $0 \leq r, s \leq 3$, the monomial $x^r y^s$ is a linear combination of the ψ_{ij}.

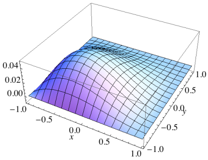
Geometry:



ψ_{11}



ψ_{21}



ψ_{22}

$$u = u|_{(0,0)}\psi_{11} + \partial_x u|_{(0,0)}\psi_{21} + \partial_y u|_{(0,0)}\psi_{12} + \partial_x\partial_y u|_{(0,0)}\psi_{22} + \cdots, \qquad \forall u \in \mathcal{Q}_3([0,1]^2)$$

Which monomials do we really *need* for cubic order?

tensor product cubics (dim=16)

$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

total degree cubics (dim=10)

superlinear cubics (dim=12)

$$\text{total degree}(x^r y^s) = r + s$$

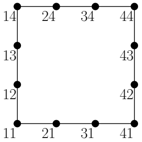
$$\text{superlinear degree}(x^r y^s) = r + s - \{\# \text{ of linearly appearing variables}\}$$

	total degree	superlinear degree
xy^2	3	2
x^3y	4	3
xy^3	4	3
x^2y^2	4	4
x^3y^2	5	5

→ For cubic order accuracy, we only *need* all total degree cubics.
→ To ensure a "smooth enough" solution, we expand to the set of all superlinear degree cubics.
→ The notion of superlinear degree and its generalization for serendipity elements comes from ARNOLD, AWANOU Found. Comp Math 2011, Math. Comp. 2013.

Cubic *serendipity* basis functions in 2D and 3D

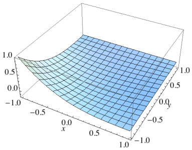
$$\underbrace{\left\{ \begin{array}{c} x^r y^s \\ \text{sldeg} \leq 3 \end{array} \right\}}_{\mathcal{S}_3([0,1]^2)} \longleftrightarrow \left\{ \begin{array}{c} \vartheta_{\ell m}(x,y) \\ \text{(limited indices)} \end{array} \right\} \longleftrightarrow$$



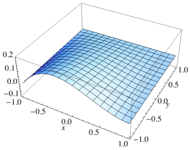
monomials $\longleftrightarrow$ **basis functions** $\longleftrightarrow$ **domain points**

Approximation: For sldeg$(x^r y^s) \leq 3$, $x^r y^s$ is a linear combination of the $\vartheta_{\ell m}$.

Geometry:



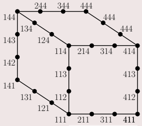
ϑ_{11}



ϑ_{21}

$$\begin{aligned} a(x,y) &= a|_{(0,0)}\vartheta_{11} \\ &+ \partial_x a|_{(0,0)}\vartheta_{21} \\ &+ \partial_y a|_{(0,0)}\vartheta_{12} \\ &+ \cdots \end{aligned}$$

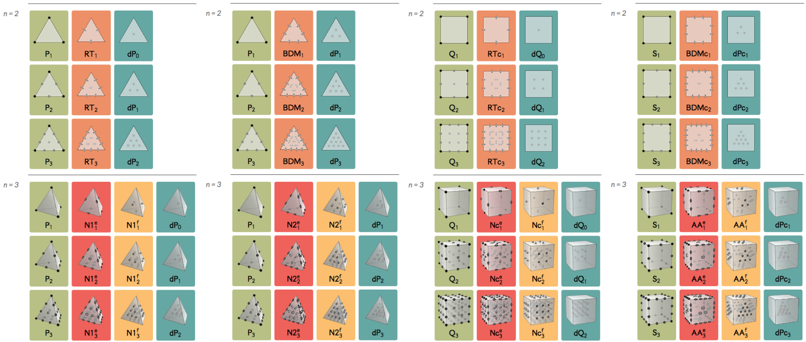
in 3D: $\left\{ \begin{array}{c} x^r y^s z^t \\ \text{sldeg} \leq 3 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \vartheta_{\ell mn} \\ \text{(limited indices)} \end{array} \right\} \longleftrightarrow$



Outline

The ‘Periodic Table of the Finite Elements’

ARNOLD, LOGG, “Periodic table of the finite elements,” *SIAM News*, 2014.



Classification of many common conforming finite element types.

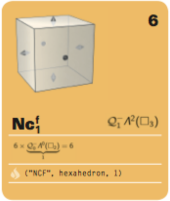
$n \rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)
$r \rightarrow$ Order $1, 2, 3$ of error decay (going down columns)
$k \rightarrow$ Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).

Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

$n \quad \rightarrow \quad$ the spatial dimension of the domain
$r \quad \rightarrow \quad$ the order of error decay
$k \quad \rightarrow \quad$ the differential form order of the solution space



Ex: $\mathcal{Q}_1^-\Lambda^2(\square_3)$ is an element for

$n = 3 \quad \rightarrow \quad$ domains in $\mathbb{R}^3$
$r = 1 \quad \rightarrow \quad$ linear order of error decay
$k = 2 \quad \rightarrow \quad$ conformity in $\Lambda^2(\mathbb{R}^3) \rightsquigarrow H(\text{div})$

$\mathcal{Q}_1^-\Lambda^2(\square_3)$ is part of the $\mathcal{Q}^-$ 'column' of elements,
is defined on geometry $\square_3$ (i.e. a cube),
has a **6** dimensional space of test functions,
and has an associated set of **6** degrees of freedom
that are unisolvent for the test function space.

An abbreviated reading list (50 years of theory!)

RAVIART, THOMAS, "A mixed finite element method for 2nd order elliptic problems" *Lecture Notes in Mathematics*, 1977 ← *3172 citations, including 150 from 2017!*

NÉDÉLEC, "Mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1980

BREZZI, DOUGLAS JR., MARINI, "Two families of mixed finite elements for second order elliptic problems," *Numerische Mathematik*, 1985

NÉDÉLEC, "A new family of mixed finite elements in $\mathbb{R}^3$," *Numerische Mathematik*, 1986

ARNOLD, FALK, WINTHER "Finite element exterior calculus, homological techniques, and applications," *Acta Numerica*, 2006

CHRISTIANSEN, "Stability of Hodge decompositions in finite element spaces of differential forms in arbitrary dimension," *Numerische Mathematik*, 2007

ARNOLD, FALK, WINTHER "Finite element exterior calculus: from Hodge theory to numerical stability," *Bulletin of the AMS*, 2010

ARNOLD, AWANOU "The serendipity family of finite elements ", *Found. Comp Math*, 2011

ARNOLD, AWANOU "Finite element differential forms on cubical meshes", *Math Comp.*, 2013

ARNOLD, BOFFI, BONIZZONI "Finite element differential forms on curvillinear meshes and their approximation properties," *Numerische Mathematik*, 2014

$H(\text{div})$ / L^2 mixed form of Poisson problem

Derivation of a mixed method for the **Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

Given $f : \Omega \to \mathbb{R}$, find a function $p \in H^2(\Omega)$ such that

$$\Delta p + f = 0, \qquad \text{in } \Omega, + \text{B.C.'s}$$

Writing this as a first order system: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} \text{div } \mathbf{u} + f &= 0, && \text{in } \Omega, \\ \mathbf{u} - \text{grad } p &= 0, && \text{in } \Omega, \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (\text{div } \mathbf{u}, w) + (f, w) &= 0, && \forall \ w \in L^2 = \Lambda^3(\Omega) \\ (\mathbf{u}, \mathbf{v}) + (p, \text{div } \mathbf{v}) &= 0, && \underbrace{\forall \ \mathbf{v} \in H(\text{div})}_{\text{i.e. } \mathbf{v},\ \text{div } \mathbf{v} \in L^2(\Omega)} = \underbrace{\Lambda^2(\Omega)}_{\text{differential form notation}} \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda^2_h$ and $p \in \Lambda^3_h$ such that

$$\begin{aligned} (\text{div } \mathbf{u}_h, w_h) + (f, w_h) &= 0 && \forall \ w_h \in \Lambda^3_h \subset L^2(\Omega) \\ (\mathbf{u}_h, \mathbf{v}_h) + (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall \ \mathbf{v}_h \in \Lambda^2_h \subset H(\text{div}) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A conforming mixed method for Darcy Flow

Movement of a fluid through porous media modeled via **Darcy flow**:

Given f and g, find pressure p and velocity $\mathbf{u}$ such that:

$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 && \text{in } \Omega, \\ \text{div } \mathbf{u} - f &= 0 && \text{in } \Omega, \\ p &= g && \text{on } \partial\Omega, \end{aligned}$$

where K is a symmetric, uniformly positive definite tensor for $\dfrac{\text{permeability}}{\text{viscosity}}$.

A **weak form** of these equations: find $\mathbf{u} \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v}) - (p, \text{div } \mathbf{v}) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v} \in H(\text{div}) \\ (\text{div } \mathbf{u}, w) - (f, w) &= 0 && \forall\, w \in L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

A conforming mixed **finite element** method: find $\mathbf{u}_h \in \Lambda_h^2$ and $p \in \Lambda_h^3$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v}_h \in \Lambda_h^2 && \subset H(\text{div}) \\ (\text{div } \mathbf{u}_h, w_h) - (f, w_h) &= 0 && \forall\, w_h \in \Lambda_h^3 && \subset L^2(\Omega) \\ (\partial\Omega \text{ conditions}) &= 0 \end{aligned}$$

ARBOGAST, PENCHEVA, WHEELER, YOTOV "A Multiscale Mortar Mixed Finite Element Method" *Multiscale Modeling and Simulation* (SIAM) 6:1, 2007.

Stable pairs of finite element spaces

$$
\begin{aligned}
(\mathbf{u}_h, \mathbf{v}_h) + (p_h, \text{div } \mathbf{v}_h) &= [\partial\Omega \text{ terms}] && \forall\, \mathbf{v}_h \in \Lambda_h^2 && \subset H(\text{div}) \\
(\text{div } \mathbf{u}_h, w_h) + (f, w_h) &= 0 && \forall\, w_h \in \Lambda_h^3 && \subset L^2(\Omega)
\end{aligned}
$$

Given a selection for the finite element spaces $(\Lambda_h^2, \Lambda_h^3)$,

the method is said to be **stable** if the error in the computed solution $(\mathbf{u}_h, p_h)$ is within a constant multiple C of the minimal *possible* error. That is:

$$
||\mathbf{u} - \mathbf{u}_h||_{H(\text{div})} + ||p - p_h||_{L^2} \leq C \left(\inf_{\mathbf{w}\in\Lambda_h^2} ||\mathbf{u} - \mathbf{w}||_{H(\text{div})} + \inf_{q\in\Lambda_h^3} ||p - q||_{L^2} \right) \quad (*)
$$

Brezzi's theorem establishes the following sufficient criteria for $(*)$:

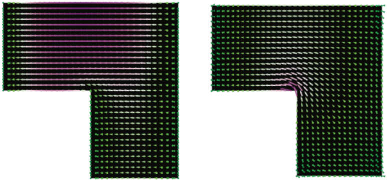
$$
(\mathbf{w}, \mathbf{w}) \geq c\,||\mathbf{w}||^2_{H(\text{div})}, \qquad \forall\, \mathbf{w} \in \mathbf{Z}_h := \left\{ \mathbf{w} \in \Lambda_h^2 \;:\; (\text{div } \mathbf{w}, q) = 0, \;\; \forall\, q \in \Lambda_h^3 \right\},
$$

$$
\sup_{\mathbf{w}\in\Lambda_h^2} \frac{(\text{div } \mathbf{w}, q)}{||\mathbf{w}||_{H(\text{div})}} \geq c\,||q||_{L^2}, \qquad \forall\, q \in \Lambda_h^3.
$$

If the pair $(\Lambda_h^2, \Lambda_h^3)$ satisfies these two criteria it is called a **stable pair**.

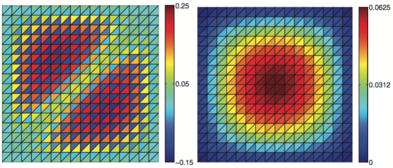
BREZZI, "On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers," RAIRO, 1974.

The importance of method selection



Vector Poisson problem

- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.



Poisson problem

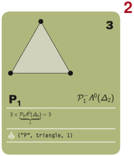
- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution: $x(1-x)y(1-y)$.

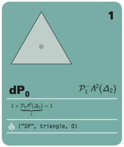
Examples and images borrowed from:

ARNOLD, FALK, WINTHER "Finite Element Exterior Calculus: From Hodge Theory to Numerical Stability," *Bulletin of the AMS*, 47:2, 2010.

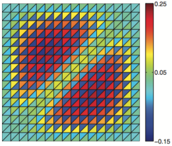
Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

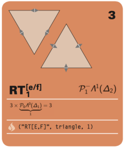


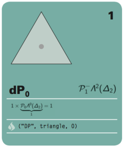


$\subset H^1 \times H^1$ $\quad$ $\subset L^2$ $\quad \Rightarrow$

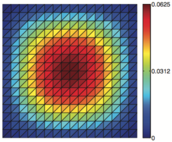


Unstable method





$\subset H(\text{div})$ $\quad$ $\subset L^2$ $\quad \Rightarrow$



Provably stable method

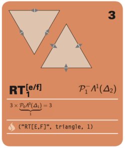
converges to

$u = x(1-x)y(1-y)$

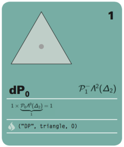
Example and images on right from:

ARNOLD, FALK, WINTHER "Finite Element Exterior Calculus. . ." *Bulletin of the AMS*, 47:2, 2010.

Method selection and cochain complexes

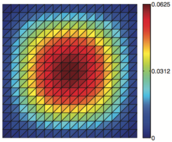


$\subset H(\mathrm{div})$



$\subset L^2$

$\Rightarrow$



Provably stable method

converges to
$\mathrm{u} = x(1-x)y(1-y)$

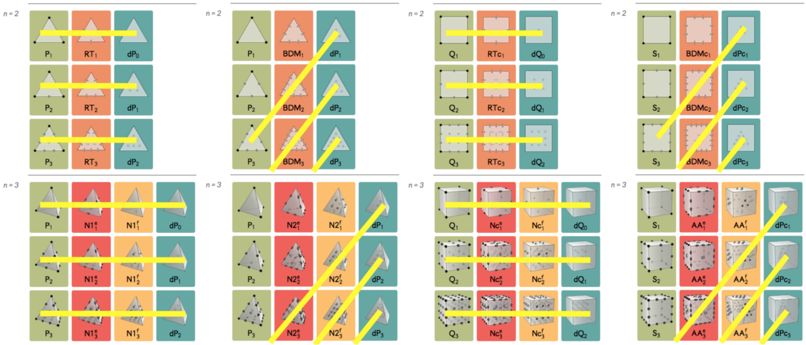
Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the L^2 deRham Diagram.

$$H^1 \xrightarrow[\text{grad}]{\nabla} H(\mathrm{curl}) \xrightarrow[\text{curl}]{\nabla\times} H(\mathrm{div}) \xrightarrow[\text{div}]{\nabla\cdot} L^2$$

	H^1	$H(\mathrm{curl})$	$H(\mathrm{div})$	L^2
vector Poisson	σ	μ		
Maxwell's eqn's		h	b	
Darcy / Poisson			$\mathbf{u}$	p

Stable pairs are found from consecutive entries in a cochain complex.

Exact cochain complexes found in the table



- *Sequences* of elements are used to design stable mixed methods for problems like Darcy flow, Maxwell's equations, vector Poisson, etc.
- The sequences occur either horizontally or diagonally in the table as shown.

Exact cochain complexes found in the table

On an n-simplex in $\mathbb{R}^n$:

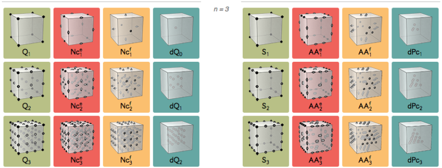
$$\mathcal{P}_r^-\Lambda^0 \to \mathcal{P}_r^-\Lambda^1 \to \cdots \to \mathcal{P}_r^-\Lambda^{n-1} \to \mathcal{P}_r^-\Lambda^n$$ **'trimmed' polynomials**

$$\mathcal{P}_r\Lambda^0 \to \mathcal{P}_{r-1}\Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1}\Lambda^{n-1} \to \mathcal{P}_{r-n}\Lambda^n$$ **polynomials**

On an n-dimensional cube in $\mathbb{R}^n$:

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n$$ **tensor product**

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n$$ **serendipity**

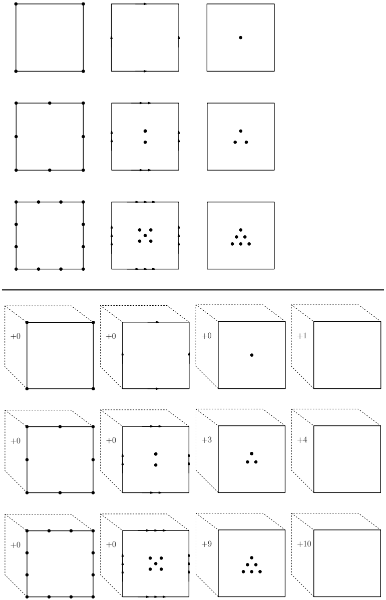


The 'minus' spaces proceed across rows of the PToFE (***r* is fixed**) while the 'regular' spaces proceed along diagonals (***r* decreases**)

Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.

Outline

The 5th column: Trimmed serendipity spaces



A new column for the PToFE:
the **trimmed serendipity** elements.

$\boxed{\mathcal{S}_r^- \Lambda^k(\square_n)}$ denotes

- approximation order r,
- subset of k-form space $\Lambda^k(\Omega)$,
- use on meshes of n-dim'l cubes.

Defined for any $n \geq 1$, $0 \leq k \leq n$, $r \geq 1$

Identical or analogous properties to all the other colummns in the table.

The advantage of the $\mathcal{S}_r^- \Lambda^k$ spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.

Key properties of the trimmed serendipity spaces

$$\mathcal{Q}_r^-\Lambda^0 \to \mathcal{Q}_r^-\Lambda^1 \to \cdots \to \mathcal{Q}_r^-\Lambda^{n-1} \to \mathcal{Q}_r^-\Lambda^n \qquad \textbf{tensor product}$$

$$\mathcal{S}_r\Lambda^0 \to \mathcal{S}_{r-1}\Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1}\Lambda^{n-1} \to \mathcal{S}_{r-n}\Lambda^n \qquad \textbf{serendipity}$$

$$\mathcal{S}_r^-\Lambda^0 \to \mathcal{S}_r^-\Lambda^1 \to \cdots \to \mathcal{S}_r^-\Lambda^{n-1} \to \mathcal{S}_r^-\Lambda^n \qquad \textbf{trimmed serendipity}$$

Subcomplex: $d\mathcal{S}_r^-\Lambda^k \subset \mathcal{S}_r^-\Lambda^{k+1}$

Exactness: The above sequence is exact.
i.e. the image of incoming map = kernel of outgoing map

Inclusion: $\mathcal{S}_r\Lambda^k \subset \mathcal{S}_{r+1}^-\Lambda^k \subset \mathcal{S}_{r+1}\Lambda^k$

Trace: $\mathrm{tr}_f\mathcal{S}_r^-\Lambda^k(\mathbb{R}^n) \subset \mathcal{S}_r^-\Lambda^k(f)$, for any $(n-1)$-hyperplane f in $\mathbb{R}^n$

Special cases:
$\mathcal{S}_r^-\Lambda^0 = \mathcal{S}_r\Lambda^0$
$\mathcal{S}_r^-\Lambda^n = \mathcal{S}_{r-1}\Lambda^n$
$\mathcal{S}_r^-\Lambda^k + d\mathcal{S}_{r+1}\Lambda^{k-1} = \mathcal{S}_r\Lambda^k$.

Replace '$\mathcal{S}$' by '$\mathcal{P}$' $\rightsquigarrow$ key properties about the first two columns for $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$!

Dimension count and comparison

Formula for counting degrees of freedom of $\mathcal{S}_r^-\Lambda^k(\square_n)$:

$$\sum_{d=k}^{\min\{n,\lfloor r/2\rfloor+k\}} 2^{n-d}\binom{n}{d}\left(\binom{r-d+2k-1}{r-d+k-1}\binom{r-d+k-1}{d-k}+\binom{r-d+2k}{k}\binom{r-d+k-1}{d-k-1}\right)$$

	k	r=1	2	3	4	5	6	7
n=2	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65
	2	1	3	6	10	15	21	28
n=3	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360
	2	6	21	45	82	135	207	301
	3	1	4	10	20	35	56	84
n=4	0	16	48	80	136	216	328	480
	1	32	112	216	392	656	1036	1563
	2	24	96	216	422	746	1227	1910
	3	8	36	94	200	375	644	1036
	4	1	5	15	35	70	126	210

Mixed Method dimension comparison 1

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order r decay in the approximation of p, $\mathbf{u}$, and div $\mathbf{u}$ is desired.

Total # of degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1\rvert + \lvert\mathcal{Q}_r^-\Lambda^2\rvert$	$\lvert\mathcal{S}_r\Lambda^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1\rvert + \lvert\mathcal{S}_r^-\Lambda^2\rvert$
1	4+1 = 5	8+1 = 9	4+1 = 5
2	12+4 = 16	14+3 = 17	10+3 = 13
3	24+9 = 33	22+6 = 28	17+6 = 23

Total # of degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2\rvert + \lvert\mathcal{Q}_r^-\Lambda^3\rvert$	$\lvert\mathcal{S}_r\Lambda^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2\rvert + \lvert\mathcal{S}_r^-\Lambda^3\rvert$
1	6+1 = 7	18+1 = 19	6+1 = 7
2	36+8 = 44	39+4 = 43	21+4 = 25
3	108+27 = 135	72+10 = 82	45+10 = 55

Mixed Method dimension comparison 2

Mixed method for Darcy problem: $\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

of **interior** degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda_0^1\rvert + \lvert\mathcal{Q}_r^-\Lambda_0^2\rvert$	$\lvert\mathcal{S}_r\Lambda_0^1\rvert + \lvert\mathcal{S}_{r-1}\Lambda_0^2\rvert$	$\lvert\mathcal{S}_r^-\Lambda_0^1\rvert + \lvert\mathcal{S}_r^-\Lambda_0^2\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	4+4 = 8	2+3 = 5	2+3 = 5
3	12+9 = 21	6+6 = 12	5+6 = 11

of **interior** degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda_0^2\rvert + \lvert\mathcal{Q}_r^-\Lambda_0^3\rvert$	$\lvert\mathcal{S}_r\Lambda_0^2\rvert + \lvert\mathcal{S}_{r-1}\Lambda_0^3\rvert$	$\lvert\mathcal{S}_r^-\Lambda_0^2\rvert + \lvert\mathcal{S}_r^-\Lambda_0^3\rvert$
1	0+1 = 1	0+1 = 1	0+1 = 1
2	12+8 = 20	3+4 = 7	3+4 = 7
3	54+27 = 81	12+10 = 22	9+10 = 19

Mixed Method dimension comparison 3

Mixed method for Darcy problem:
$$\begin{aligned} \mathbf{u} + K\nabla p &= 0 \\ \text{div } \mathbf{u} - f &= 0 \end{aligned}$$

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements *still* have the fewest DoFs:

of **interface** (edge) degrees of freedom on a square ($n = 2$):

r	$\lvert\mathcal{Q}_r^-\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r\Lambda^1(\partial\square_2)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^1(\partial\square_2)\rvert$
1	4	8	4
2	8	12	8
3	12	16	12

of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

r	$\lvert\mathcal{Q}_r^-\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r\Lambda^2(\partial\square_3)\rvert$	$\lvert\mathcal{S}_r^-\Lambda^2(\partial\square_3)\rvert$
1	6	18	6
2	24	36	18
3	54	60	36

Outline

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Goal: find a computational basis for $\mathcal{S}_1\Lambda^1(\square_3)$:

- Must be H(curl)-conforming
- Must have 24 functions, 2 associated to each edge of cube
- Must recover constant and linear approx. on each edge
- The approximation space contains:

  (1) Any polynomial coefficient of at most linear order:
  $\{1, x, y, z\} \times \{dx, dy, dz\} \;\to$ 12 forms

  (2) Certain forms with quadratic or cubic order coefficients shown in table at left $\to$ 12 forms

- For constants, use "obvious" functions:

$$\{(y \pm 1)(z \pm 1)dx,\;\; (x \pm 1)(z \pm 1)dy,\;\; (x \pm 1)(y \pm 1)dz\}$$

e.g. $(y+1)(z+1)dx$ evaluates to zero on every edge ***except*** $\{y=1, z=1\}$ where it is $\equiv 4 \to$ constant approx.

Also, $(y+1)(z+1)dx$ can be written as linear combo, by using the first three forms at left to get the $yz\ dx$ term

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

- For constant approx on edges, we used:

$$\{(y\pm1)(z\pm1)dx,\ \ (x\pm1)(z\pm1)dy,\ \ (x\pm1)(y\pm1)dz\}$$

- Guess for linear approx on edges:

$$\{x(y\pm1)(z\pm1)dx,\ \ y(x\pm1)(z\pm1)dy,\ \ z(x\pm1)(y\pm1)dz\}$$

e.g. $x(y+1)(z+1)dx$ evaluates to $4x$ on $\{y=1, z=1\}$.

- Unfortunately: $\boxed{x(y+1)(z+1)dx \notin \mathcal{S}_1\Lambda(\square_3)!}$

Why? $x(y+1)(z+1)dx = (xyz + xy + xz + x)dx$

but *xyz dx only appears with other cubic order coefficients!*

- Remedy: add *dy* and *dz* terms that vanish on all edges.

Building a computational basis

dx	dy	dz
$-yz$	xz	0
0	$-xz$	xy
yz	xz	xy
$2xy$	x^2	0
$2xz$	0	x^2
y^2	$2xy$	0
0	$2yz$	y^2
z^2	0	$2xz$
0	z^2	$2yz$
$2xyz$	x^2z	x^2y
y^2z	$2xyz$	xy^2
yz^2	xz^2	$2xyz$

Computational basis element associated to $\{y = 1, z = 1\}$:

$$\boxed{2x(y+1)(z+1)\ dx + (z+1)(x^2-1)\ dy + (y+1)(x^2-1)\ dz}$$

- ✓ Evaluates to $4x$ on $\{y = 1, z = 1\}$ (linear approx.)
- ✓ Evaluates to 0 on all other edges
- ✓ Belongs to the space $\mathcal{S}_1\Lambda(\square_3)$:

$$\begin{array}{rcrcrl} 2xyz\ dx & + & x^2z\ dy & + & x^2y\ dz & \\ 2xy\ dx & + & x^2\ dy & + & 0\ dz & \\ 2xz\ dx & + & 0\ dy & + & x^2\ dz & \\ 2x\ dx & + & (-z-1)dy & + & (-y-1)dz & \leftarrow \text{linear order} \\ \hline \end{array}$$

$\hookrightarrow$ summation and factoring yields the desired form)

There are 11 other such functions, one per edge. We have:

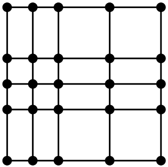
$$\mathcal{S}_1\Lambda(\square_3) = \underbrace{E_0\Lambda^1(\square_3)}_{\text{"obvious" basis for constant approx}} \oplus \underbrace{\tilde{E}_1\Lambda^1(\square_3)}_{\text{modified basis for linear approx}}$$

$$\dim\ 24 = 12 + 12$$

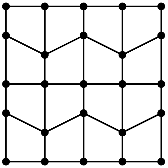
A complete table of computational bases

$n=3$	$k=0$	$k=1$	$k=2$	$k=3$
$\mathcal{S}_r\Lambda^k$	$V\Lambda^0(\square_3)$	$\emptyset$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$	$\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3) \oplus \tilde{E}_r\Lambda^1(\square_3)$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$	$\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \hat{F}_r\Lambda^1(\square_3)$	$\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3) \oplus \tilde{F}_r\Lambda^2(\square_3)$	$\emptyset$
	$\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\bigoplus_{i=4}^{r} I_i\Lambda^1(\square_3)$	$\bigoplus_{i=2}^{r} I_i\Lambda^2(\square_3)$	$\bigoplus_{i=2}^{r} I_i\Lambda^3(\square_3)$
$\mathcal{S}_r^-\Lambda^k$	$V\Lambda^0(\square_3)$	$\emptyset$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=0}^{r-2} E_i\Lambda^0(\square_3)$	$\bigoplus_{i=0}^{r-1} E_i\Lambda^1(\square_3)$	$\emptyset$	$\emptyset$
	$\bigoplus_{i=4}^{r} F_i\Lambda^0(\square_3)$	$\bigoplus_{i=2}^{r-1} F_i\Lambda^1(\square_3) \oplus \tilde{F}_r\Lambda^1(\square_3)$	$\bigoplus_{i=0}^{r-1} F_i\Lambda^2(\square_3)$	$\emptyset$
	$\bigoplus_{i=6}^{r} I_i\Lambda^0(\square_3)$	$\bigoplus_{i=4}^{r-1} I_i\Lambda^1(\square_3) \oplus \tilde{I}_r\Lambda^1(\square_3)$	$\bigoplus_{i=2}^{r-1} I_i\Lambda^2(\square_3) \oplus \tilde{I}_r\Lambda^2(\square_3)$	$\bigoplus_{i=2}^{r-1} I_i\Lambda^3(\square_3)$

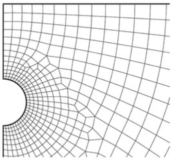
From squares and cubes to quads and hexes



structured mesh with affinely mapped quad elements



structured mesh with non-affinely mapped quad elements



unstructured mesh with non-affinely mapped quads; image from Zhao, Yu, Tao 2013

Ideally: construct bases on unstructured quad meshes by passing basis functions through a non-affine geometry transformation.

Unfortunately: Non-affine maps of serendipity-type elements result in sub-optimal convergence rates (see e.g. Arnold, Boffi, Falk 2002).

A way out: Adjust the mapping procedure and adjust some basis functions to the physical element (see e.g. Arbogast, Correa 2016)

Potential impact: A complete conforming finite element theory that can be applied in software for generic quad/hex meshes (consider: CUBIT, `deal.ii`, FEniCS, etc)

Acknowledgments

Thanks for the invitation to speak!

Related Publications

G., Kloefkorn "Trimmed Serendipity Finite Element Differential Forms." Mathematics of Computation, to appear. See `arXiv:1607.00571`

G., Kloefkorn, Sanders "Computational serendipity and tensor product finite element differential forms." Submitted. See `arXiv:1806.00031`

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Slides and Pre-prints

`http://math.arizona.edu/~agillette/`