Quality Meshing of a Forest of Branching Structures

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joint work with

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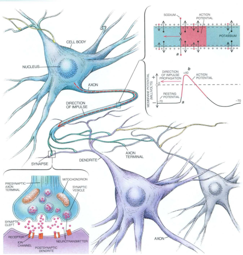
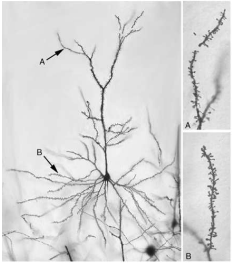
Outline

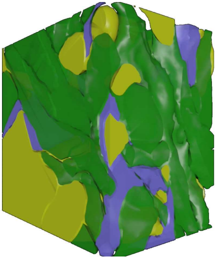
Outline

Motivation: Neuronal Modeling

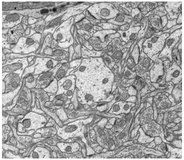


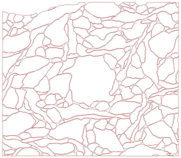
How Neurons Communicate

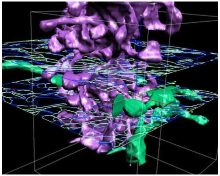




Neurons in vivo are packed very densely and have many small geometric features known to affect voltage decay.





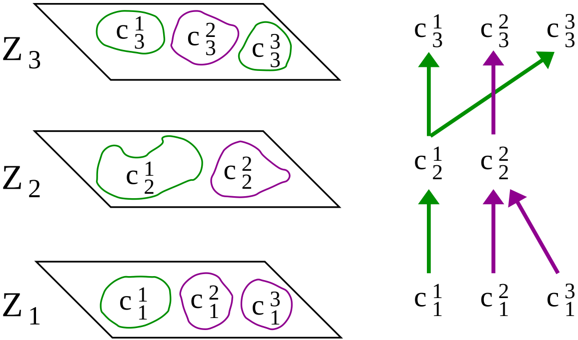


Neuron length $\sim$ 100 μm; Neuropil dataset $\sim$ 2 $(\mu m)^3$; in-plane resolution $\sim$ 5-10 nm

Formal Problem Statement

Input (for a K component forest):

1. M horizontal planes $Z_1, \ldots Z_M$. (Z_m given by $z = z_m$)
2. K functions $g_m^1, \ldots, g_m^K : \mathbb{R}^2 \to \mathbb{R}$ such that $\bigcup_{k=1}^{K} \{g_m^k = 0\}$ is a 1-manifold.
3. Contours $\{c_m^k\}$ of the set $\{g_m^k = 0\}$.
4. An acyclic directed graph G with vertices $\{c_m^k\}$, indicating connectivity.



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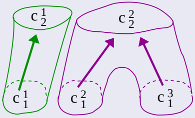
Output K functions $h_1, \ldots, h_K : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that:

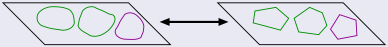
1. Each h_k restricts to g_m^k on Z_m, i.e. $h_k(x, y, z_m) \equiv g_m^k(x, y)$.
2. Each surface $h_k(x, y, z) = 0$ is a compact, connected, smooth 2-manifold with local connectivity corresponding to the graph G.
3. The K component surface $\prod_{k=1}^{K} h_k(x, y, z) = 0$ is a 2-manifold, i.e. the component surfaces do not intersect.



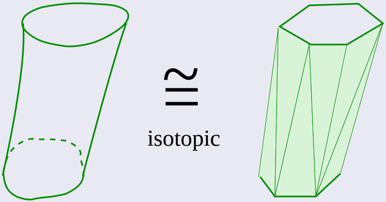
Simplifying Assumptions

Assumptions for Reduction to Meshing:

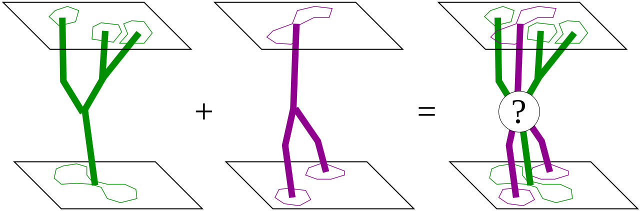
1. Contours are simple polygons and can be refined if necessary.



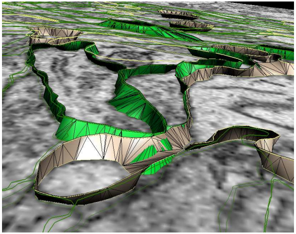
2. A mesh of the polygonal contours satisfying the output properties is isotopic to a smooth solution.

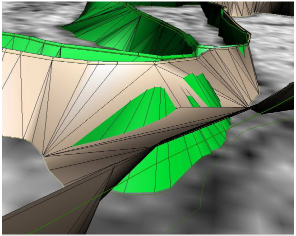


The Multi-Component Difficulty



Independent solutions to the reconstruction problem for each component may produce topological or geometrical inaccuracies when aggregated.





Outline

Selected Prior Work on Single Component Problem

FUCHS, KEDEM AND USELTON *Optimal surface reconstruction from planar contours* Communications of the ACM 20:10, 1977.

- Seminal work in reconstruction from polygonal contours.

MEYERS, SKINNER AND SLOAN *Surfaces from contours* ACM Transactions on Graphics 11:3, 1992.

- Identified subproblems of correspondence, tiling, and branching.

BAREQUET AND SHARIR *Piecewise-linear interpolation between polygonal slices* Symposium on Computational Geometry 1994.

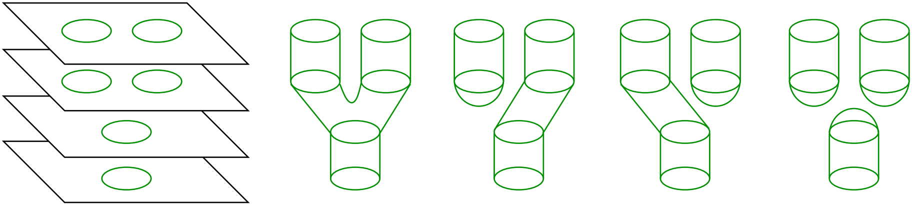
- Developed an algorithm for CT, MRI, and other medical applications.

BAJAJ, COYLE AND LIN *Arbitrary topology shape reconstruction from planar cross sections* Graphic Models and Image Processing 58:6, 1996.

- Expanded this algorithm by providing topological guarantees on the output.
- We use this method for our approach.

The Correspondence Problem

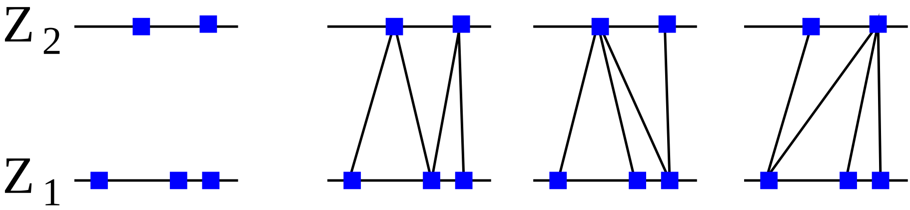
How should contours on adjacent slices connect?



Remark: This is resolved by the connectivity graph G in our case.

The Tiling Problem

How should corresponding contours be tiled?



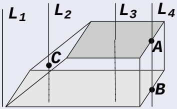
Definition: A ***slice chord*** is an edge connecting vertices on adjacent slices. A ***tiling triangle*** is formed by two slice chords and a contour edge.

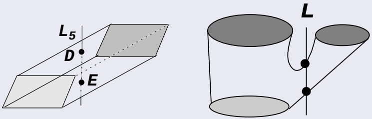
What are the criteria of a "good" tiling?

The Tiling Problem

Desired Tiling Criteria

1. The reconstructed surface forms a piecewise closed surface of polyhedra.
2. Any vertical line segment between two adjacent slices intersecting the reconstructed surface does so at exactly one point or along exactly one line segment.

OK: 

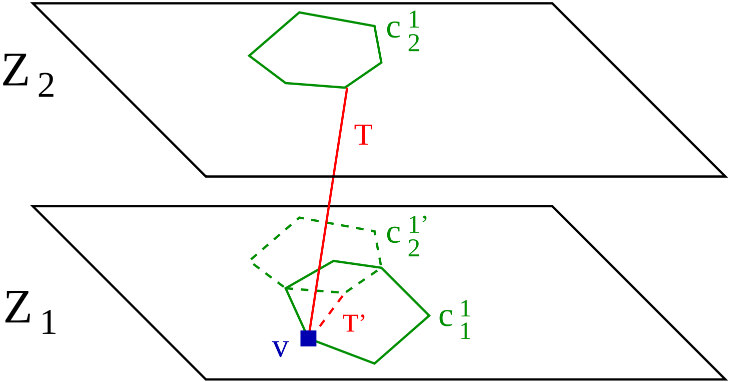
not OK: 

3. Resampling of the reconstructed surface on any slice reproduces the original contours.

Remark: The last criterion implies that aside from contour refinement, any edges or vertices added to the mesh must lie outside the Z_i planes.

The Tiling Problem Resolved

- Let v be a vertex in contour $c_1^1 \subset Z_1$ corresponding to $c_2^1 \subset Z_2$.
- Let T be a slice chord from v to c_2^1.
- Let the "prime" notation denote vertical projection to Z_1.

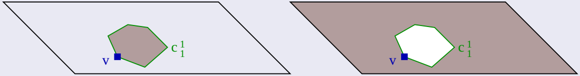


The Tiling Problem Resolved

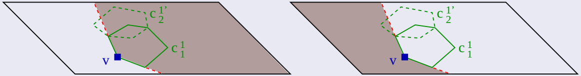
Theorem *[Bajaj, Coyle, Lin 1996]*:

If a tiling satisfies the three criteria, the following hold:

i) T' lies in exactly one of these regions:



ii) If $v \notin c_2^{1'}$ then T' lies in exactly one of these regions:



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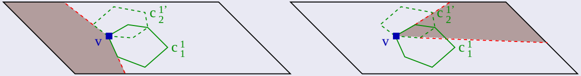


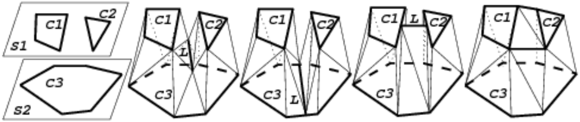
The Tiling Problem Resolved

Tiling Algorithm *[Bajaj, Coyle, Lin 1996]*:

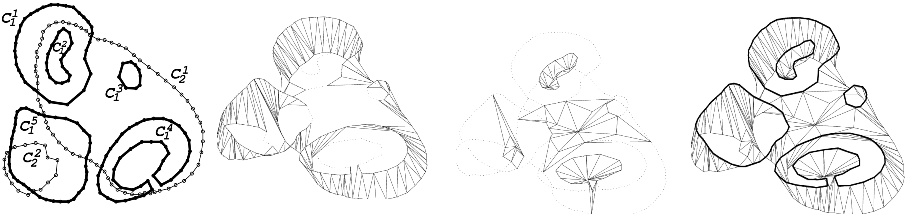
1. For each vertex $v \in \{c_1^j\}$, make a list of all the slice chords that could be formed to a vertex of $\{c_2^k\}$ (based on the resolution of the correspondence problem).
2. Select the shortest length chord from this list which satisfies the results of the Theorem.
3. If no chord from the list satisfies the theorem, tag the vertex as "untiled."
4. Collect boundaries of untiled regions for subsequent meshing when resolving the branching problem.

The Branching Problem

How should tiling be done when a contour in Z_1 corresponds to more than one contour in Z_2?

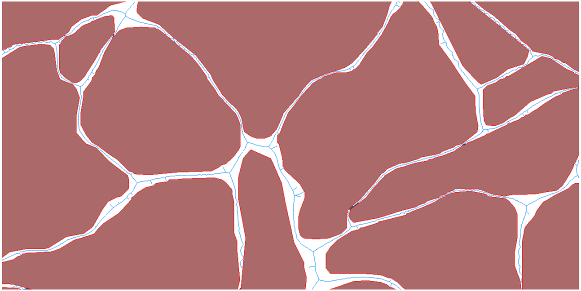


To ensure the criteria are sastisfied, we add vertices to a plane half way between Z_1 and Z_2 and then mesh.



The Medial Axis

For our approach, we will use the medial axis of the region exterior to the contours on each slice.



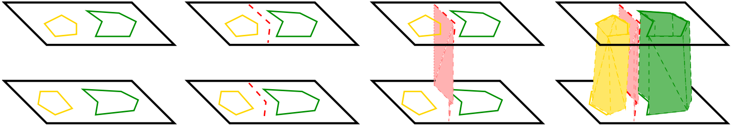
Definition: The ***medial axis*** $\mathfrak{M}$ of an open set $O \subset \mathbb{R}^n$ is the set of points $x \in O$ for which there are at least two closest points to x on the complement O^c.

Outline

1. Motivation and Problem Statement
2. Prior Work and Background
3. Approach
4. Initial Results

Algorithm

Idea: Compute the medial axis on each slice, scaffold it to a three dimensional partitioning, and use this to detect and remove intersections between components.



MULTISURFRECON$\left(\{Z_i\}, \left\{c_i^j\right\}, \{P_i\}, G\right)$ (P_i = vertices of the contours c_i^j)

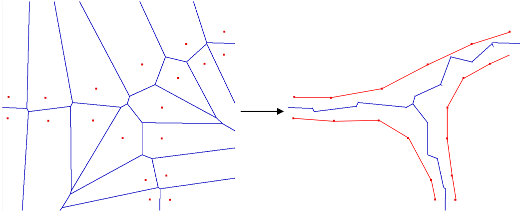
1. Construct and partition medial axes on each slice.
2. Construct a medial surface.
3. Create single component surfaces.
4. Remove overlaps on intermediate planes.
5. Improve mesh quality.

For subsequent explanation, suppose G has only two components, yellow and green, so that each c_i^j is either a yellow contour (YC) or green contour (GC).

Algorithm

1. Construct and partition medial axes on each slice.

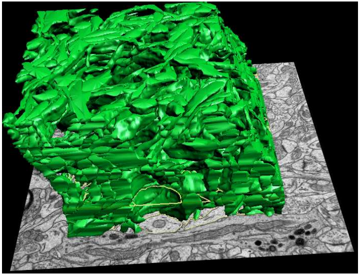
- Compute the Voronoi diagram Vor P_i of the sample points P_i of the contours on slice i.
- Discard all Voronoi edges intersecting the contours.
- Discard all remaining vertices and edges interior to the contours.
- The remaining vertices and edges approximate the medial axis $\mathfrak{M}$ of extracellular space.
- Let $E_{GY} = \{$edges of $\mathfrak{M}$ defined by a point of YC and a point of GC$\}$

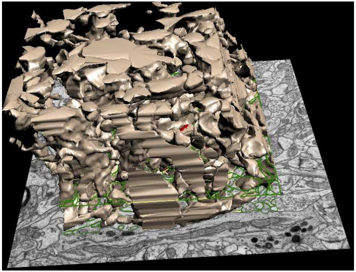


2. Construct a medial surface.

Dilate the edges in E_{GY} so that they form contours and color these contours red. Create a partial, approximate medial surface $\mathfrak{M}S$ by running the SINGLESURFRECON code on E_{GY}.

3. Create single component surfaces.

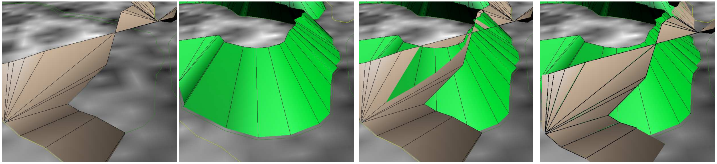




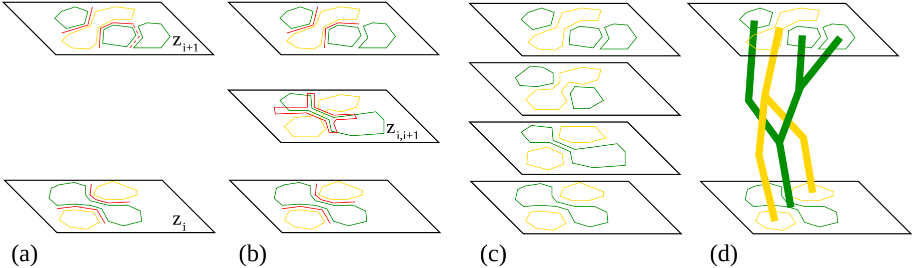
Algorithm

4. Remove overlaps on intermediate planes.

Simple overlap:

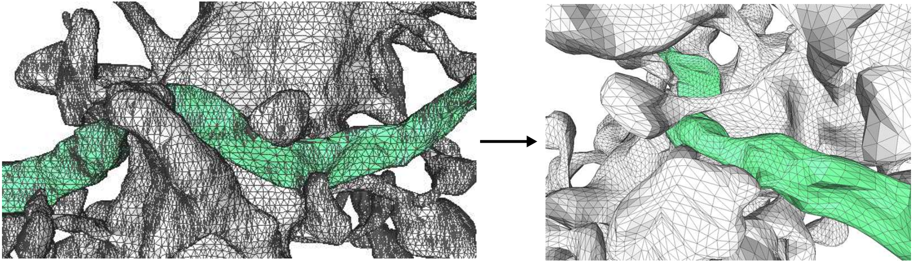


Exotic overlap:



Algorithm

5. Improve mesh quality.



- Decimation: QSlim software by Michael Garland.
- Shape Improvement: Geometric Flow library from LBIE Mesher, part of Volume Rover software by CVC lab.

Correctness Results

Lemma 1:

Suppose that for each i, there are no pairwise overlaps between the original contours on plane Z_i. Then the output of SINGLESURFRECON run on any subset of the original contours is not self-intersecting.

Proof: This is a consequence of the three tiling criteria; if the original contours do not overlap, SINGLESURFRECON cannot create self-intersections.

Lemma 2:

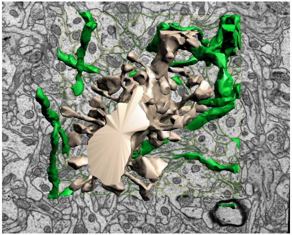
If there are no overlaps of any of the intermediate planes where branching occurs, then the output of MULTISURFRECON has no self-intersections in the entire volume. Conversely, if the output has no self-intersections in the volume, it does not have any overlap on any intermediate plane.

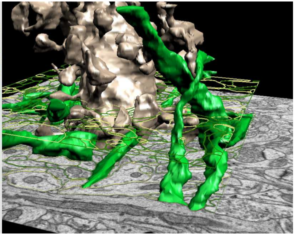
Proof: $\Rightarrow$: The output is a linear interpolation between consecutive planes. If the contours do not overlap, their interpolation does not intersect.

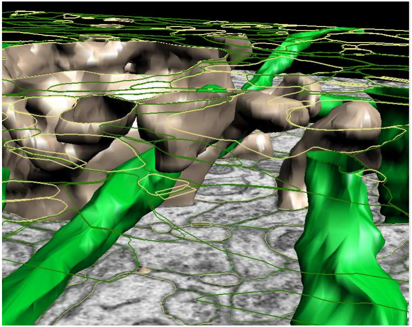
$\Leftarrow$: If the output is not self-intersecting, the components cannot intersect on any plane.

Outline

Initial Results







Acknowledgements

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