Generalized Barycentric Coordinates for Polygonal Finite Elements

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joint work with Chandrajit Bajaj and Alexander Rand

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Why consider polygonal finite elements?

- **Theoretical:** Discrete Exterior Calculus considerations

- **Applied:** A new approach to longstanding meshing problems
  - Sliver removal by local remeshing
  - Canonical adaptive meshing elements

- **Practical:** Generic approach to coding would encompass old and new methods.
Overview of Approach

linear elements: $\{\lambda_i\}$ = (triangular) barycentric coordinates

higher order elements $\{\lambda_i \lambda_j\}$

vector elements $\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\}$

linear elements: $\{\lambda_i\} = \textit{generalized}$ barycentric coordinates
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1. Types of Generalized Barycentric Coordinates
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Definition

Let $\Omega$ be a convex polygon in $\mathbb{R}^2$ with vertices $v_1, \ldots, v_n$. Functions $\lambda_i : \Omega \rightarrow \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on $\Omega$ if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on $\Omega$.

2. **Linear Completeness**: For any linear function $L : \Omega \rightarrow \mathbb{R}$, $L = \sum_{i=1}^{n} L(v_i)\lambda_i$.

Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity**: $\sum_{i=1}^{n} \lambda_i \equiv 1$.

4. **Linear precision**: $\sum_{i=1}^{n} v_i\lambda_i(x) = x$.

5. **Interpolation**: $\lambda_i(v_j) = \delta_{ij}$.

**Theorem [Warren, 2003]**

If the $\lambda_i$ are rational functions of degree $n - 2$, then they are unique.
Many generalizations to choose from . . .

- Wachspress

- Sibson

- Harmonic

- Mean value

Many more in graphics contexts...
Triangulation Coordinates

Let $\mathcal{T}$ be a triangulation of $\Omega$ formed by adding edges between the $v_j$ in some fashion. Define

$$\lambda_{i,\mathcal{T}} : \Omega \to \mathbb{R}$$

to be the barycentric function associated to $v_i$ on triangles in $\mathcal{T}$ containing $v_i$ and identically 0 otherwise. Trivially, these are barycentric coordinates on $\Omega$.

Theorem [Floater, Hormann, Kós, 2006]

For a fixed $i$, let $\mathcal{T}_m$ denote any triangulation with an edge between $v_{i-1}$ and $v_{i+1}$. Let $\mathcal{T}_M$ denote the triangulation formed by connecting $v_i$ to all the other $v_j$. Any barycentric coordinate function $\lambda_i$ satisfies the bounds

$$0 \leq \lambda_{i,\mathcal{T}_m}(x) \leq \lambda_i(x) \leq \lambda_{i,\mathcal{T}_M}(x) \leq 1, \quad \forall x \in \Omega.$$
Harmonic Coordinates

Let $g_i : \partial \Omega \to \mathbb{R}$ be the piecewise linear function satisfying

$$g_i(v_j) = \delta_{ij}, \quad g_i \text{ linear on each edge of } \Omega.$$

The **harmonic coordinate** function $\lambda_i^{\text{Har}}$ is defined to be the solution of Laplace’s equations with $g_i$ as boundary data,

$$\begin{cases} 
\Delta (\lambda_i^{\text{Har}}) = 0, & \text{on } \Omega, \\
\lambda_i^{\text{Har}} = g_i, & \text{on } \partial \Omega.
\end{cases}$$

These coordinates are **optimal** in the sense that they minimize the norm of the gradient over all functions satisfying the boundary conditions:

$$\lambda_i^{\text{Har}} = \arg\min \left\{ |\lambda|_{H^1(\Omega)} : \lambda = g_i \text{ on } \partial \Omega \right\}.$$
Wachspress Coordinates

Let \( \mathbf{x} \in \Omega \) and define \( A_i(\mathbf{x}) \) and \( B_i \) as the areas shown.

Define the Wachspress weight function as

\[
\lambda_{i}^{\text{Wach}}(\mathbf{x}) = \frac{w_{i}^{\text{Wach}}(\mathbf{x})}{\sum_{j=1}^{n} w_{j}^{\text{Wach}}(\mathbf{x})}
\]

The Wachspress coordinates are then given by the *rational* functions

\[
w_{i}^{\text{Wach}}(\mathbf{x}) = B_i \prod_{j \neq i, i-1} A_j(\mathbf{x}).
\]
Sibson (Natural Neighbor) Coordinates

Let $P$ denote the set of vertices $\{v_i\}$ and define $P' = P \cup \{x\}$.

$$C_i := |V_P(v_i)| = |\{y \in \Omega : |y - v_i| < |y - v_j|, \forall j \neq i\}|$$

= area of cell for $v_i$ in Voronoi diagram on the points of $P$,

$$D(x) := |V_{P'}(x)| = |\{y \in \Omega : |y - x| < |y - v_i|, \forall i\}|$$

= area of cell for $x$ in Voronoi diagram on the points of $P'$.

By a slight abuse of notation, we also define

$$D(x) \cap C_i := |V_{P'}(x) \cap V_P(v_i)|.$$

The Sibson coordinates are defined to be

$$\lambda_{i}^{\text{Sibs}}(x) := \frac{D(x) \cap C_i}{D(x)}$$

or, equivalently,

$$\lambda_{i}^{\text{Sibs}}(x) = \frac{D(x) \cap C_i}{\sum_{j=1}^{n} D_j(x) \cap C_j}.$$
1. Types of Generalized Barycentric Coordinates
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Let $\Omega$ be a convex polygon with vertices $v_1, \ldots, v_n$.

For linear elements, an **optimal convergence estimate** has the form

\[
\left\| u - \sum_{i=1}^{n} u(v_i)\lambda_i \right\|_{H^1(\Omega)} \leq C \text{diam}(\Omega) \| u \|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega).
\]  

(1)

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in $H^1$ norm.


For (1), it suffices to prove an $H^1$-interpolant estimate over domains of diameter one:

\[
\left\| \sum_{i=1}^{n} u(v_i)\lambda_i \right\|_{H^1(\Omega)} \leq C_i \| u \|_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega).
\]  

(2)

For (2), it suffices to bound the gradients of the $\{\lambda_i\}$, i.e. prove $\exists \ C_\lambda \in \mathbb{R}$ such that

\[
\| \nabla \lambda_i \|_{L^2(\Omega)} \leq C_\lambda.
\]  

(3)
Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need estimates of the geometry.

Let $\rho(\Omega)$ denote the radius of the largest inscribed circle. The **aspect ratio** $\gamma$ is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$

Three possible geometric conditions on a polygonal mesh

- **G1. Bounded aspect ratio:** There exists $\gamma^* < \infty$ such that
  $$\gamma < \gamma^*$$

- **G2. Minimum edge length:** There exists $d_* > 0$ such that
  $$|\mathbf{v}_i - \mathbf{v}_{i-1}| > d_*$$

- **G3. Maximum interior angle:** There exists $\beta^* < \pi$ such that
  $$\beta_i < \beta^*$$
Summary of convergence results

**Theorem**

In the table below, any necessary geometric criteria to achieve the optimal convergence estimate are denoted by N. The set of geometric criteria denoted by S in each row are sufficient to guarantee estimate.


<table>
<thead>
<tr>
<th></th>
<th>G1 aspect ratio</th>
<th>G2 min. edge</th>
<th>G3 max angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangulated</td>
<td>$\lambda^{\text{Tri}}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Wachspress</td>
<td>$\lambda^{\text{Wach}}$</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>Sibson</td>
<td>$\lambda^{\text{Sibs}}$</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>Harmonic</td>
<td>$\lambda^{\text{Har}}$</td>
<td>S</td>
<td>-</td>
</tr>
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</table>
Implication of convergence results

- **GEOMETRIC CONSTRAINTS**
  - Triangulate
  - Wachspress
  - Sibson
  - Harmonic

- **EFFICIENT IMPLEMENTATION**
  - Triangulate
  - Wachspress
  - Sibson
  - Harmonic
Outline

1. Types of Generalized Barycentric Coordinates
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From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:

\[ \{ \lambda_i \} \xrightarrow{\text{pairwise products}} \{ \lambda_a \lambda_b \} \]

Why is this naïve?

- For an \( n \)-gon, this construction gives \( n + \binom{n}{2} \) basis functions \( \lambda_a \lambda_b \)
- The space of quadratic polynomials is only dimension 6: \( \{1, x, y, xy, x^2, y^2\} \)
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge \( \Rightarrow \text{only } 2n \text{ functions needed!} \)

Problem Statement

Construct \( 2n \) basis functions associated to the vertices and edge midpoints of an arbitrary \( n \)-gon such that a quadratic convergence estimate is obtained.
For quadrilaterals, the ‘serendipity’ element for rectangles has long been known to provide quadratic convergence.


The technique works more generally for affine mappings of the reference element to a physical element (‘affine’ = preserves collinearity and ratios of distances)

For non-affine meshes of quadrilaterals, however, the serendipity construction is known to provide sub-optimal convergence.


### Mapped biquadratic elements

<table>
<thead>
<tr>
<th>$n$</th>
<th>$|u - u_h|_{L^2}$ err.</th>
<th>$%$</th>
<th>Rate</th>
<th>$|\nabla (u - u_h)|_{L^2}$ err.</th>
<th>$%$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.5e-02</td>
<td>2.877</td>
<td></td>
<td>4.5e-01</td>
<td>37.253</td>
<td></td>
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<tr>
<td>4</td>
<td>4.4e-03</td>
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<td>3.0</td>
<td>1.1e-01</td>
<td>9.333</td>
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<tr>
<td>8</td>
<td>5.5e-04</td>
<td>0.045</td>
<td>3.0</td>
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<td>2.329</td>
<td>2.0</td>
</tr>
<tr>
<td>16</td>
<td>6.9e-05</td>
<td>0.006</td>
<td>3.0</td>
<td>7.1e-03</td>
<td>0.583</td>
<td>2.0</td>
</tr>
<tr>
<td>32</td>
<td>8.6e-06</td>
<td>0.001</td>
<td>3.0</td>
<td>1.8e-03</td>
<td>0.146</td>
<td>2.0</td>
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<tr>
<td>64</td>
<td>1.1e-06</td>
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<td>3.0</td>
<td>4.4e-04</td>
<td>0.036</td>
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<td>3.0</td>
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<td>0.047</td>
<td>2.0</td>
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### Serendipity elements

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<td>2.877</td>
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<tr>
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<td>9.333</td>
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<td>2.1</td>
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<td>0.401</td>
<td>1.1</td>
</tr>
</tbody>
</table>

**ARNOLD, BOFFI, FALK,** *Approximation by Quadrilateral Finite Elements,* 2002.
Generalized barycentric coordinates allow for a quadratic serendipity construction on any quadrilateral.

Since the analysis holds for affine mappings, these serve as reference elements for a wider range of quadrilaterals.

The trapezoidal meshes satisfy the geometry bounds and hence we can recover the optimal convergence rate.

| n  | ||u - uh||_L^2   | ||∇(u - uh)||_L^2 |
|----|----------------|------------------|
| 2  | 2.34e-3        | 2.22e-2          |
| 4  | 3.03e-4 2.95   | 6.10e-3 1.87     |
| 8  | 3.87e-5 2.97   | 1.59e-3 1.94     |
| 16 | 4.88e-6 2.99   | 4.04e-4 1.97     |
| 32 | 6.13e-7 3.00   | 1.02e-4 1.99     |
| 64 | 7.67e-8 3.00   | 2.56e-5 1.99     |
| 128| 9.59e-9 3.00   | 6.40e-6 2.00     |
| 256| 1.20e-9 3.00   | 1.64e-6 1.96     |

Rand, Gillette, Bajaj  *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011
We define matrices $\mathbf{A}$ and $\mathbf{B}$ to reduce the naïve quadratic basis.

**filled dot** = Lagrangian domain point

= all functions in the set evaluate to 0 except the associated function which evaluates to 1

**open dot** = non-Lagrangian domain point

= partition of unity satisfied, but not Lagrange property

$$\begin{align*}
\{\lambda_i\} \quad &\text{pairwise products} \quad \rightarrow \quad \{\mu_{ab}\} \quad A \quad \rightarrow \quad \{\xi_{ij}\} \quad B \quad \rightarrow \quad \{\psi_{ij}\}
\end{align*}$$

$\lambda$ = Linear

$\mu$ = Quadratic

$\xi$ = Serendipity

$\psi$ = Lagrange
Serendipity basis functions $\xi_{ij}$ are constructed as a linear combination of pairwise product functions $\mu_{ab}$:

$$[\xi_{ij}] = A \begin{bmatrix} \frac{\mu_{aa}}{\mu_{a(a+1)}} \\ \frac{\mu_{a(a+1)}}{\mu_{ab}} \end{bmatrix} = [I \ C_{ij}] \begin{bmatrix} \frac{\mu_{aa}}{\mu_{a(a+1)}} \\ \frac{\mu_{a(a+1)}}{\mu_{ab}} \end{bmatrix}$$

The quadratic basis is ordered as follows:

- $\mu_{aa} =$ basis functions associated with vertices
- $\mu_{a(a+1)} =$ basis functions associated with edge midpoints
- $\mu_{ab} =$ basis functions associated with interior diagonals, i.e. $b \notin \{a - 1, a, a + 1\}$

- The first two types are left alone, resulting in the identify matrix above.
- The $c_{ij}$ values define how the interior basis functions are added into the boundary basis functions.
We require the serendipity basis to have quadratic approximation power:

- **Constant precision (CP):** \( \sum_i \xi_{ii} + 2\xi_{i(i+1)} = 1. \)

- **Linear precision (LP):** \( \sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)} = \mathbf{x}. \)

- **Quadratic precision (QP):** \( \sum_i \mathbf{v}_i \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)} = \mathbf{x}\mathbf{x}^T. \)

- Six constraints (CP, LP, QP) \( \Rightarrow \) six non-zero \( c_{ab}^{ij} \) per column.

- We select (arbitrarily) that \( \mu_{ab} \) contributes to \( \xi_{a,a}, \xi_{b,b}, \) and their neighbors.

**Theorem**

Constants \( \{c_{ij}^{ab}\} \) exist for any convex polygon such that the resulting basis \( \{\xi_{ij}\} \) satisfies the CP, LP, and QP requirements.
Pairwise products vs. Lagrange basis

Pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:

Translation between these two bases is straightforward and generalizes to the higher dimensional case...
From serendipity to Lagrange

\{\xi_{ij}\} \xrightarrow{B} \{\psi_{ij}\}

\[
\begin{bmatrix}
\psi_{11} \\
\psi_{22} \\
\vdots \\
\psi_{nn} \\
\psi_{12} \\
\vdots \\
\psi_{1n}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 \\
-1 & -1 & \ddots & \ddots \\
-1 & -1 \\
0 & 4 & 4 \\
& & & \ddots \\
& & & & 4
\end{bmatrix}
\begin{bmatrix}
\xi_{11} \\
\xi_{22} \\
\vdots \\
\xi_{nn} \\
\xi_{12} \\
\vdots \\
\xi_{1n}
\end{bmatrix}
= B[\xi_{ij}].
Serendipity Theorem

Given bounds on polygon aspect ratio (G1), minimum edge length (G2), and maximum interior angles (G3):

- $||A||$ is uniformly bounded,
- $||B||$ is uniformly bounded, and
- The basis $\{\psi_{ij}\}$ interpolates smooth data with $O(h^2)$ error.

**Theorem**

**Rand, Gillette, Bajaj** *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011
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From scalar to vector elements

Barycentric functions are used to define $H(\text{curl})$ vector elements on triangles:

$$\{\lambda_i\} \xrightarrow{\text{Whitney construction}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$

Generalized barycentric functions provide $H(\text{curl})$ elements on polygons:

$$\{\lambda_i\} \xrightarrow{\text{Whitney construction}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$

This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications.

**Gillette, Bajaj** *Dual Formulations of Mixed Finite Element Methods with Applications*
**Conformity:** The basis functions \( \{ \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \} \) interpolate an \( H(\text{curl}) \) function.

Let \( T_E \vec{\nu} \) denote the tangential projection of \( \vec{\nu} \) to an edge \( E \).

\[
\vec{v} \in H(\text{curl}) := \left\{ \vec{v} \in \left( L^2(\Omega) \right)^3 \text{ s.t. } \nabla \times \vec{v} \in \left( L^2(\Omega) \right)^3 \right\}
\]

\[
\vec{v} \in H(\text{curl}) \iff T_E \vec{v} \in C^0, \quad \forall \text{ edges } E \text{ in mesh}
\]

\[
\lambda_k \equiv 0 \quad \text{on } E \nsubseteq v_k
\]

\[
\therefore \nabla \lambda_k \perp E \quad \text{on } E \nsubseteq v_k
\]

\[
\therefore T_E(\lambda_i \nabla \lambda_j) \neq 0 \iff v_i, v_j \in E
\]

**Interpolation:** The basis functions are Lagrange-like for edge integrals.

\[
T_{e_{ij}}(\nabla \lambda_i) = \frac{1}{|e_{ij}|}, \quad \text{since the } \lambda_i \text{ are linear on edges.}
\]

\[
\int_{e_{ij}} (\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) \cdot \vec{e}_{ij} = \frac{1}{|e_{ij}|} \int_{e_{ij}} \lambda_i + \lambda_j = \frac{1}{|e_{ij}|} \int_{e_{ij}} 1 = 1.
\]
Future directions

Future work and open problems

- Extension to 3D generalized barycentric functions.
- Extension to 3D vector interpolation functions on polytopes.
- Implementation in a finite element solver for comparison studies.
Questions?

- Slides and pre-prints available at http://ccom.ucsd.edu/~agillette