DEC-deRham Conformity for Mixed Finite Element Methods

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joint work with

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Outline

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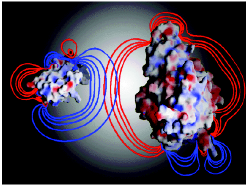
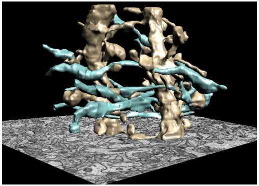
Motivation

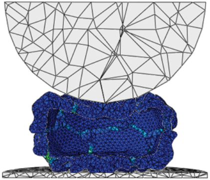
Biological modeling requires **robust** computational methods to solve PDEs



Electromagnetics



Electrodiffusion



Elasticity

These methods must accommodate

- multiple variables
- large meshes
- multi-scale phenomena

What does **robust** mean in such contexts?

Problem Statement

A robust computational method for solving PDEs should exhibit

- **Model Conformity:** Computed solutions are found in a subspace of the solution space for the continuous problem
  Criterion: Discrete solution spaces replicate the the deRham sequence.
- **Discretization Stability:** The true error between the discrete and continuous solutions is bounded by a multiple of the best approximation error
  Criterion: The discrete inf-sup condition is satisfied.
- **Bounded Roundoff Error:** Accumulated numerical errors due to machine precision do not compromise the computed solution
  Criterion: Matrices inverted by the linear solver are well-conditioned.

Problem Statement

Use the theory of Discrete Exterior Calculus to evaluate the robustness of existing computational methods for PDEs arising in biology and create novel methods with improved robustness. This talk's focus: model conformity.

Selected Prior Work

- Importance of differential geometry in computational methods for electromagnetics:

BOSSAVIT *Computational Electromagnetism* Academic Press Inc. 1998

- Primer on DEC theory and program of work:

DESBRUN, HIRANI, LEOK, MARSDEN *Discrete Exterior Calculus* arXiv:math/0508341v2 [math.DG], 2005

- Generalization of deRham diagram criteria for model conformity:

ARNOLD, FALK, WINTHER *Finite element exterior calculus, homological techniques, and applications* Acta Numerica, 15:1-155, 2006.

- Applications of DEC to electromagnetics, Darcy flow, and elasticity problems:

HE, TEIXEIRA *Geometric finite element discretization of Maxwell equations in primal and dual spaces* Physics Letters A, 349(1-4):1–14, 2006

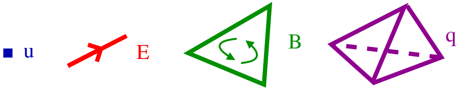
HIRANI, NAKSHATRALA, CHAUDHRY *Numerical method for Darcy Flow derived using Discrete Exterior Calculus* arXiv:0810.3434v1 [math.NA], 2008

YAVARI *On geometric discretization of elasticity* Journal of Mathematical Physics, 49(2):022901-1–36, 2008

Outline

(Smooth) Exterior Calculus

- Differential *k*-forms model *k*-dimensional physical phenomena.



- The exterior derivative *d* generalizes common differential operators.

$$\Lambda^0(\Omega) \xrightarrow[\text{grad}]{d_0} \Lambda^1(\Omega) \xrightarrow[\text{curl}]{d_1} \Lambda^2(\Omega) \xrightarrow[\text{div}]{d_2} \Lambda^3(\Omega)$$

- The Hodge Star transfers information between complementary dimensions.

$$\Lambda^0(\Omega) \longleftarrow * \longrightarrow \Lambda^3(\Omega)$$

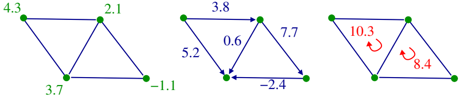
$$\Lambda^1(\Omega) \longleftarrow * \longrightarrow \Lambda^2(\Omega)$$

Fundamental "Theorem" of Discrete Exterior Calculus

Model-conforming computational methods must recreate the essential properties of (continuous) exterior calculus on the discrete level.

Discrete Exterior Calculus

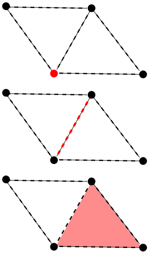
- Discrete differential k-forms are k-cochains, i.e. linear functions on k-simplices.



- The discrete exterior derivative is $\mathbb{D} = (\partial)^T$, the transpose of the boundary operator.

$$\mathcal{C}^0 \xrightarrow[\text{(grad)}]{\mathbb{D}_0} \mathcal{C}^1 \xrightarrow[\text{(curl)}]{\mathbb{D}_1} \mathcal{C}^2 \xrightarrow[\text{(div)}]{\mathbb{D}_2} \mathcal{C}^3$$

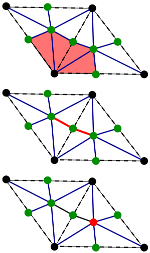
- The discrete Hodge Star $\mathbb{M}$ transfers information between complementary dimensions on **dual** meshes.



$$\mathcal{C}^0 \xrightarrow{\mathbb{M}_0} \overline{\mathcal{C}}^2$$

$$\mathcal{C}^1 \xrightarrow{\mathbb{M}_1} \overline{\mathcal{C}}^1$$

$$\mathcal{C}^2 \xrightarrow{\mathbb{M}_2} \overline{\mathcal{C}}^0$$



The Importance of Cohomology

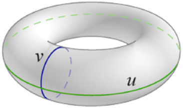
$$\Lambda^0 \xrightarrow[\text{grad}]{d_0} \Lambda^1 \xrightarrow[\text{curl}]{d_1} \Lambda^2 \xrightarrow[\text{div}]{d_2} \Lambda^3$$

$$\mathcal{C}^0 \xrightarrow{\mathbb{D}_0} \mathcal{C}^1 \xrightarrow{\mathbb{D}_1} \mathcal{C}^2 \xrightarrow{\mathbb{D}_2} \mathcal{C}^3$$

Cohomology classes represent the different types of solutions permitted by the topology of the space.

The solution spaces for a discrete method should include representatives from all cohomology classes. Hence **model conformity** requires that the top and bottom sequences have the same cohomology group ranks.

Example: The torus has two non-zero cohomology equivalence classes in dim. 1



$$\begin{aligned} \dim(\text{Cohomology at } \Lambda^1) &:= \dim(\ker d_1 / \text{im } d_0) \\ \| \ (\text{if conforming}) & \\ \dim(\text{Cohomology at } \mathcal{C}^1) &:= \dim(\ker \mathbb{D}_1 / \text{im } \mathbb{D}_0) \end{aligned}$$

Two Notions of "Interpolant"

In classical finite element theory, a **local interpolant operator** I associated to a finite element $\{K, P, \Sigma\}$ is a map from a normed vector space $V(K) \supset P$ to P.

Ex: The local interpolant operator for the linear Lagrange element on the tetrahedron K from $V(K) = (C^0)^3$ is

$$I : (C^0)^3 \to H^1(K), \qquad \nu \mapsto \sum_{i=1}^{4} \nu(v_i)\lambda_i$$

where λ_i is the barycentric function on the tetrahedron for vertex v_i.

For DEC, we define an **interpolation map** $\mathcal{I}_k$ as a map from k-cochains $\mathcal{C}^k$ to differential k-forms Λ^k.

Ex: The interpolant map for 0-cochains on a tetrahedron is

$$\mathcal{I}_0 : \mathcal{C}^0 \to H^1(K), \qquad \omega \mapsto \sum_{i=1}^{4} \omega(v_i)\lambda_i$$

Note that $\nu : K \to \mathbb{R}$ while $\omega : \{v_i\} \to \mathbb{R}$.

Mixed finite element methods

Mixed finite element methods seek solutions in subspaces of the L^2 deRham sequence.

$$\begin{array}{ccccccc} H^1 & \xrightarrow[\text{grad}]{d_0} & H(\text{curl}) & \xrightarrow[\text{curl}]{d_1} & H(\text{div}) & \xrightarrow[\text{div}]{d_2} & L^2 \\ \mathcal{I}_0 \uparrow \downarrow \mathcal{P}_0 & & \mathcal{I}_1 \uparrow \downarrow \mathcal{P}_1 & & \mathcal{I}_2 \uparrow \downarrow \mathcal{P}_2 & & \mathcal{I}_3 \uparrow \downarrow \mathcal{P}_3 \\ \mathcal{C}^0 & \xrightarrow{\mathbb{D}_0} & \mathcal{C}^1 & \xrightarrow{\mathbb{D}_1} & \mathcal{C}^2 & \xrightarrow{\mathbb{D}_2} & \mathcal{C}^3 \end{array}$$

where $\mathcal{I}$ is an interpolation map and $\mathcal{P}$ is a projection map (the deRham map).

Theorem [Arnold, Falk, Winther]

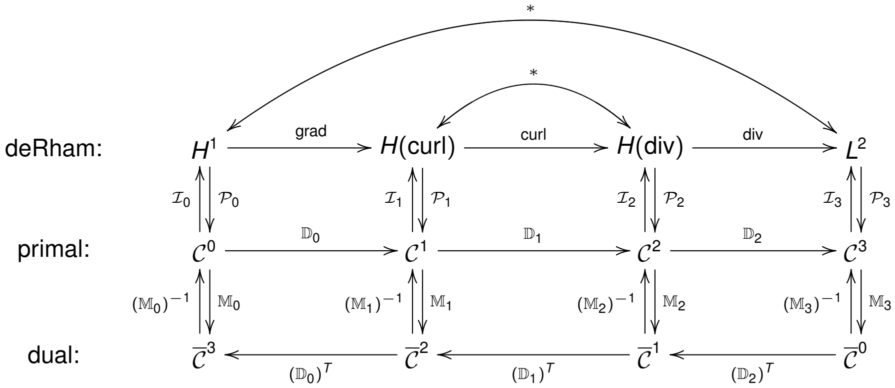
If $\mathcal{I}_k$ is Whitney interpolation and $\mathcal{P}_{k+1} d_k = \mathbb{D}_k \mathcal{P}_k$ then the top and bottom sequences have **isomorphic** cohomology.

Proof: The cohomology induced by Whitney interpolation is the simplicial cohomology [Whitney 1957] which is isomorphic to the deRham cohomology [deRham]. □

Whitney interpolation provides for model conformity in simple cases.

The DEC-deRham Diagram for $\mathbb{R}^3$

We combine the Discrete Exterior Calculus maps with the L^2 deRham sequence.



The combined diagram helps elucidate primal and dual formulations of finite element methods.

Darcy Flow in $\mathbb{R}^3$ - Primal Flux

$$
\begin{cases}
\vec{f} + \frac{k}{\mu}\nabla p &= 0 \quad \text{in } \Omega, \\
\text{div}\vec{f} &= \phi \quad \text{in } \Omega, \\
\vec{f}\cdot\hat{n} &= \psi \quad \text{on } \partial\Omega,
\end{cases}
$$

- $k, \mu \in \mathbb{R}$; no external body force; p.w. smooth $\Gamma := \partial\Omega$; $\int_\Omega \phi d\Omega = \int_{\partial\Omega} \psi d\Gamma$
- $\vec{f} \in \mathcal{C}^2$ is the volumetric flux through faces of the **primal** mesh
- $p \in \overline{\mathcal{C}}^0$ is the pressure at vertices of the **dual** mesh

Mixed (primal + dual) discretization:

$$
\begin{bmatrix} -(\mu/k)\mathbb{M}_2 & \mathbb{D}_2^T \\ \mathbb{D}_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{f} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix}.
$$

$$
\begin{array}{ccc}
\vec{f} & \xrightarrow{\mathbb{D}_2} & \mathbb{D}_2\vec{f} \\
\Big\downarrow \mathbb{M}_2 & & \\
\begin{array}{c}\mathbb{M}_2\vec{f} \\ (\mathbb{D}_2)^T p\end{array} & \xleftarrow[(\mathbb{D}_2)^T]{} & p
\end{array}
$$

Ref: Hirani, Nakshatrala, Chaudhry, 2008

Darcy Flow in $\mathbb{R}^3$ - Dual Flux

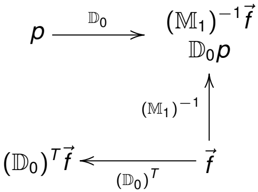
$$\begin{cases} \vec{f} + \frac{k}{\mu}\nabla p &= 0 & \text{in } \Omega, \\ \operatorname{div}\vec{f} &= \phi & \text{in } \Omega, \\ \vec{f}\cdot\hat{n} &= \psi & \text{on } \partial\Omega, \end{cases}$$

An equally valid discretization is as follows:

- $\vec{f} \in \overline{\mathcal{C}}^2$ is the volumetric flux through faces of the **dual** mesh
- $p \in \mathcal{C}^0$ is the pressure at vertices of the **primal** mesh

New mixed discretization:

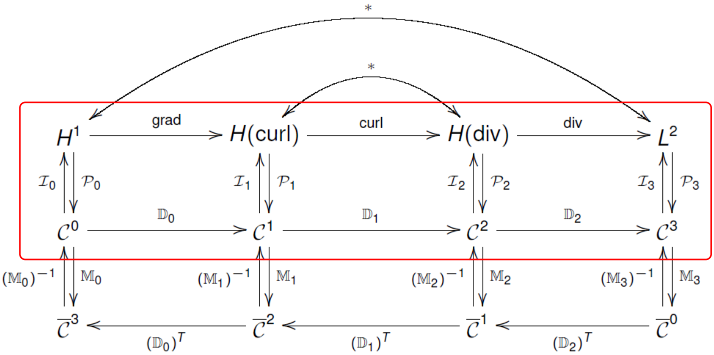
$$\begin{bmatrix} -(\mu/k)\mathbb{M}_1^{-1} & \mathbb{D}_0 \\ (\mathbb{D}_0)^T & 0 \end{bmatrix} \begin{bmatrix} \vec{f} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix}.$$



Outline

Conformity Criteria for Dual Variables

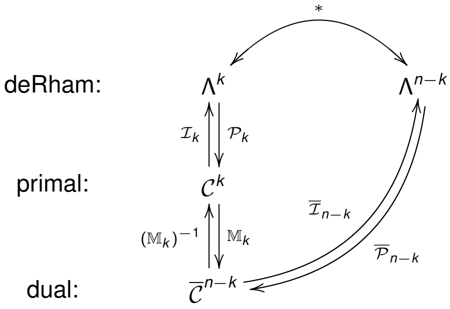
The Arnold-Falk-Winther model conformity criteria only considers primal discretizations:



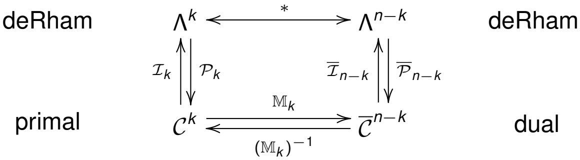
DEC-based mixed finite element methods require additional model conformity criteria.

Stability Criteria for Dual Variables

If we have projection to or interpolation from a dual mesh, we have the maps:



More concisely, we expect some commutativity of the diagram:



Stability Criteria for Dual Variables

$$
\begin{array}{ccccc}
\text{deRham} & \Lambda^k & \xleftrightarrow{\;\;*\;\;} & \Lambda^{n-k} & \text{deRham} \\
 & \mathcal{I}_k \uparrow\downarrow \mathcal{P}_k & & \overline{\mathcal{I}}_{n-k} \uparrow\downarrow \overline{\mathcal{P}}_{n-k} & \\
\text{primal} & \mathcal{C}^k & \underset{(\mathbb{M}_k)^{-1}}{\overset{\mathbb{M}_k}{\rightleftarrows}} & \overline{\mathcal{C}}^{n-k} & \text{dual}
\end{array}
$$

We identify four "subcommutativity" conditions:

$$
\begin{array}{lrcl}
\text{Commutativity at } \Lambda^k: & \mathbb{M}_k \mathcal{P}_k & = & \overline{\mathcal{P}}_{n-k} * \\
\text{Commutativity at } \mathcal{C}^k: & * \mathcal{I}_k & = & \overline{\mathcal{I}}_{n-k} \mathbb{M}_k \\
\text{Commutativity at } \Lambda^{n-k}: & (\mathbb{M}_k)^{-1} \overline{\mathcal{P}}_{n-k} & = & \mathcal{P}_k * \\
\text{Commutativity at } \overline{\mathcal{C}}^{n-k}: & \mathcal{I}_k (\mathbb{M}_k)^{-1} & = & * \overline{\mathcal{I}}_{n-k}
\end{array}
$$

To evaluate these conditions, we must now define the various maps involved.

Continuous Hodge Star

The **continuous Hodge star** is defined as the unique map $* : \Lambda^k \to \Lambda^{n-k}$ satisfying the property

$$\alpha \wedge *\beta = (\alpha, \beta)_{\Lambda^k}\, \mu, \quad \forall \alpha, \beta \in \Lambda^k$$

- $\wedge$ denotes the wedge product
- $(\cdot, \cdot)_{\Lambda^k}$ denotes the inner product on k-forms
- μ is the volume n-form on the domain

Example 1: In $\mathbb{R}^3$, let $\alpha = \beta = dx$. Then

$$\alpha \wedge *\beta = dx \wedge *dx = dx \wedge dydz = \mu = (dx, dx)_{\Lambda^k}\mu = (\alpha, \beta)_{\Lambda^k}\, \mu$$

Example 2: In $\mathbb{R}^3$, let $\alpha = dx, \beta = dy$. Then

$$\alpha \wedge *\beta = dx \wedge *dy = dx \wedge (-dxdz) = 0 = (dx, dy)_{\Lambda^k}\mu = (\alpha, \beta)_{\Lambda^k}\, \mu$$

Whitney Interpolation Map

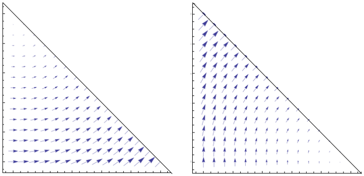
The **Whitney k-form** η_{σ^k} is associated to the k-simplex σ^k in the primal mesh.

$$\begin{array}{ll}
\sigma^0 := [v_i] & \eta_{\sigma^0} := \lambda_i \\
\sigma^1 := [v_i, v_j] & \eta_{\sigma^1} := \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i \\
\sigma^2 := [v_i, v_j, v_k] & \eta_{\sigma^2} := 2\left(\lambda_i \nabla \lambda_j \times \nabla \lambda_k + \lambda_j \nabla \lambda_k \times \nabla \lambda_i + \lambda_k \nabla \lambda_i \times \nabla \lambda_j\right) \\
\sigma^3 := [v_i, v_j, v_k, v_l] & \eta_{\sigma^3} := \chi_{\sigma^3} = \begin{cases} 1 & \text{on } \sigma^3 \\ 0 & \text{otherwise} \end{cases}
\end{array}$$

where λ_i denotes the barycentric function for vertex v_i.

The **Whitney interpolation map** $\mathcal{I}_k$ of a k-cochain ω, is

$$\mathcal{I}_k(\omega) := \sum_{\sigma^k \in \mathcal{C}_k} \omega(\sigma^k) \eta_{\sigma^k}.$$



Examples of Whitney 1-forms associated to horizontal and vertical edges, respectively

Commutativity at $\mathcal{C}^k$

$$\begin{aligned} &\text{Commutativity at } \mathcal{C}^k\text{:} & *\mathcal{I}_k &= \overline{\mathcal{I}}_{n-k}\mathbb{M}_k \\ &\text{Continuous Hodge star:} & \alpha \wedge *\beta &= (\alpha, \beta)_{\Lambda^k}\, \mu, \quad \forall \alpha, \beta \in \Lambda^k \\ &\text{Whitney interpolation map:} & \mathcal{I}_k(\omega) &= \sum_{\sigma^k \in \mathcal{C}_k} \omega(\sigma^k)\eta_{\sigma^k} \end{aligned}$$

It suffices to show that for any test function $\alpha \in \Lambda^k$

$$\alpha \wedge *\mathcal{I}_k = \alpha \wedge \overline{\mathcal{I}}_{n-k}\mathbb{M}_k.$$

Check on a basis $\{\omega_i^k\}$ where ω_i^k is 1 on σ_i^k and 0 on all other k-simplices:

$$\alpha \wedge *\mathcal{I}_k(\omega_i^k) = \alpha \wedge \overline{\mathcal{I}}_{n-k}(\mathbb{M}_k\omega_i^k).$$

Use the definitions of $\mathcal{I}_k$ and $*$ to derive the condition:

$$\left(\alpha, \eta_{\sigma_i^k}\right)_{\Lambda^k} \mu = \alpha \wedge \overline{\mathcal{I}}_{n-k}(\mathbb{M}_k\omega_i^k).$$

This condition motivates definitions of the dual interpolation map $\overline{\mathcal{I}}_{n-k}$ and the discrete Hodge star $\mathbb{M}_k$ that ensure model conformity.

Criteria Applied to Darcy Flow - Dual Flux

$$\begin{array}{ccc} p & \xrightarrow{\mathbb{D}_0} & \begin{array}{c}(\mathbb{M}_1)^{-1}\vec{f}\\ \mathbb{D}_0 p\end{array} \\ & & {\scriptstyle (\mathbb{M}_1)^{-1}}\Big\uparrow \\ (\mathbb{D}_0)^T\vec{f} & \xleftarrow[(\mathbb{D}_0)^T]{} & \vec{f}\end{array}$$

We check for commutativity of the pressure data, i.e. at $\mathcal{C}^0$ with $n = 3$, $k = 0$:

$$\left(\alpha, \eta_{\sigma_i^0}\right)_{H^1} \mu = \alpha \wedge \overline{\mathcal{I}}_3(\mathbb{M}_0 \omega_i^0) \quad \forall \alpha \in H^1$$

We use the Hodge star proposed by the authors of the paper

$$(\mathbb{M}_0)_{ii} := \frac{|\star \sigma_i^k|}{|\sigma_i^k|}$$

We use any dual interpolant $\overline{\mathcal{I}}_3$ mimicking Whitney forms, i.e.

$$\overline{\mathcal{I}}_3(\overline{\omega}) := \sum_{\star\sigma^0 \in \overline{\mathcal{C}}_3} \overline{\omega}(\star\sigma^0)\chi_{\star\sigma^0}$$

Criteria Applied to Darcy Flow - Dual Flux

The left side:

$$\begin{aligned}\left(\alpha, \eta_{\sigma_i^0}\right)_{H^1} \mu &= (\alpha, \lambda_i)_{H^1} \mu \\ &= \left(\int_K \alpha\lambda_i + \nabla\alpha \cdot \nabla\lambda_i\right) \mu\end{aligned}$$

The right side:

$$\begin{aligned}\alpha \wedge \overline{\mathcal{I}}_3(\mathbb{M}_0\omega_i^0) &= \alpha \wedge \sum_{\star\sigma^0 \in \overline{\mathcal{C}}_3} (\mathbb{M}_0^{Diag}\omega_i)(\star\sigma^0)\chi_{\star\sigma^0}\mu \\ &= \alpha \wedge |\star\sigma_i^0|\chi_{\star\sigma_i^0}\mu \\ &= \alpha|\star\sigma_i^0|\chi_{\star\sigma_i^0}\mu\end{aligned}$$

The condition:

$$\left(\int_K \alpha\lambda_i + \nabla\alpha \cdot \nabla\lambda_i\right) \mu = \alpha|\star\sigma_i^0|\chi_{\star\sigma_i^0}\mu \quad \forall\alpha \in H^1$$

Criteria Applied to Darcy Flow - Conclusions

Dual flux condition:

$$\left(\int_K \alpha\lambda_i + \nabla\alpha \cdot \nabla\lambda_i\right)\mu = \alpha\left|\star\sigma_i^0\right|\chi_{\star\sigma_i^0}\mu \quad \forall\alpha \in H^1$$

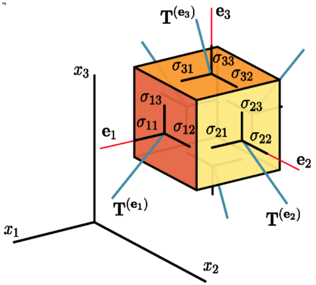
Primal flux condition:

$$\left(\int_K \alpha(x)\overline{\lambda}_i(x)\right)\mu = \alpha\left|\sigma_i^3\right|\chi_{\sigma_i^3}\mu \quad \forall\alpha \in L^2$$

- In both instances, an arbitrary test function α must be approximately constant on a neighborhood of vertex i and this constant is a multiple of a measure of the region and an integral involving α.
- This is certainly false in general, as L^2 or H^1 functions need not be locally constant.
- Hence, the diagonal Hodge star espoused by the authors does not provide a conforming method in the general setting, in either of the possible mixed finite element methods.

Outline

Elasticity Basics



(image from Wikipedia)

Elasticity problems try to find the stress σ on a domain $\Omega \subset \mathbb{R}^3$ via:

$$\begin{aligned} \text{net force (known)} &= \int_\Omega \text{body forces} \\ &= \int_{\partial\Omega} \sigma \cdot \vec{n} \\ &= \int_\Omega \text{div}\sigma \end{aligned}$$

Stress is treated as a 2-tensor since it pairs with a velocity field $\vec{v}$ and $\vec{n}$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T$$

Stress is symmetric: the σ_{ii} are normal stresses while σ_{ij} are shear stresses.

Elasticity as a PDE in $\mathbb{R}^3$ (Strong Symmetry)

Solve for stress σ and displacement u given a body force field f:

$$\begin{cases} A\sigma &= {}^{\text{sym}}\vec{\nabla} u \quad \text{in } \Omega \\ \text{div}\sigma &= f \quad \text{in } \Omega \end{cases}$$

plus boundary conditions, where

$$u \in \mathcal{V} := \text{tangent space at } x \in \Omega \cong \mathbb{R}^3$$

$$\sigma \in \mathcal{S} := \text{symmetric second order tensors}$$

The operator ${}^{\text{sym}}\vec{\nabla}$ is the symmetric gradient:

$${}^{\text{sym}}\vec{\nabla} : \mathcal{V} \to \mathcal{S}$$

$${}^{\text{sym}}\vec{\nabla} u = \frac{1}{2}\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}$$

The operator A is called a compliance tensor:

$$A : \mathcal{S} \to \mathcal{S}$$

It describes the relation between the stress σ and strain ${}^{\text{sym}}\vec{\nabla} u$.

Elasticity Complex in $\mathbb{R}^3$

$$\begin{cases} A\sigma = {}^{\text{sym}}\vec{\nabla} u & \text{in } \Omega \\ \text{div}\sigma = f & \text{in } \Omega \end{cases}$$

$$\mathcal{V} = \text{tangent space at } x \in \Omega \cong \mathbb{R}^3$$

$$\mathcal{S} = \text{symmetric second order tensors}$$

Arnold, Falk and Winther derived the following elasticity complex:

$$C^\infty(\mathcal{V}) \xrightarrow{{}^{\text{sym}}\vec{\nabla}} C^\infty(\mathcal{S}) \xrightarrow{J} C^\infty(\mathcal{S}) \xrightarrow{\text{div}} C^\infty(\mathcal{V})$$

$$u \qquad {}^{\text{sym}}\vec{\nabla} u \qquad \sigma \qquad \text{div}\sigma$$

Note that u is a V-valued 0-form while σ is a $\mathcal{S}$-valued 2-form.

We now look at how this sequence was derived.

deRham-AFW Elasticity Diagram

$$\mathcal{W} := \mathcal{V} \times \mathcal{S} \qquad d_{\mathcal{W}} := \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \qquad \pi_1, \pi_2 \text{ are surjections}$$

$$\begin{array}{ccccccc}
\Lambda^0(\mathcal{W}) & \xrightarrow{d_{\mathcal{W},0}} & \Lambda^1(\mathcal{W}) & \xrightarrow{d_{\mathcal{W},1}} & \Lambda^2(\mathcal{W}) & \xrightarrow{d_{\mathcal{W},2}} & \Lambda^3(\mathcal{W}) \\
\Phi_0^{-1} \uparrow \downarrow \Phi_0 & & \downarrow \pi_1 \circ \Phi_1 & & \downarrow \pi_2 \circ \Phi_2 & & \Phi_3^{-1} \uparrow \downarrow \Phi_3 \\
C^\infty(\mathcal{V}) & \xrightarrow{\text{sym}\,\vec{\nabla}} & C^\infty(\mathcal{S}) & \xrightarrow{J} & C^\infty(\mathcal{S}) & \xrightarrow{\text{div}} & C^\infty(\mathcal{V})
\end{array}$$

Since π_1, π_2 are surjections but not isomorphisms, we have

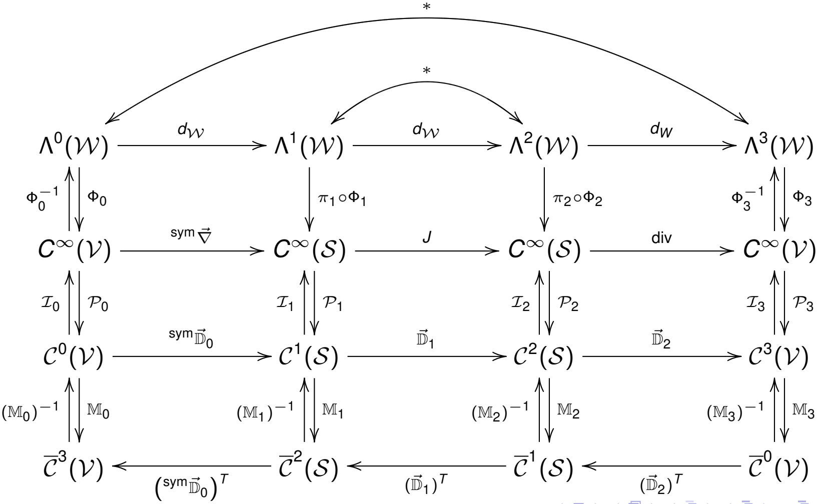
$$\dim\left(\ker J / \text{im } {}^{\text{sym}}\vec{\nabla}\right) \leq \dim\left(\ker d_{\mathcal{W},1} / \text{im } d_{\mathcal{W},0}\right)$$

$$\dim\left(\ker \text{div} / \text{im } J\right) \leq \dim\left(\ker d_{\mathcal{W},2} / \text{im } d_{\mathcal{W},1}\right)$$

Question: Under what conditions are these the inequality sharp? In other words, when is the model non-conforming?

DEC-deRham-AFW Elasticity Diagram

$\mathcal{W} := \mathcal{V} \times \mathcal{S} \qquad d_{\mathcal{W}} := \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \qquad \pi_1, \pi_2$ are surjections

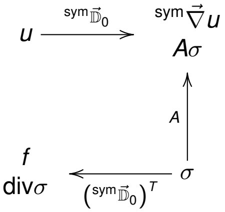


Primal-Dual Discretization

$$\begin{cases} A\sigma = {}^{\mathrm{sym}}\vec{\nabla} u & \text{in } \Omega, \\ \mathrm{div}\sigma = f & \text{in } \Omega, \end{cases}$$

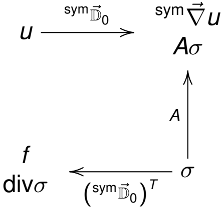
Yavari's discretization (J. Math. Physics, 2008):

- $u \in \mathcal{C}^0(\mathcal{V})$ = $\mathcal{V}$-valued primal 0-cochains
- $\sigma \in \overline{\mathcal{C}}^2$ = $\mathcal{S}$-valued dual 2-cochains



This raises a number of research directions. . .

Additional Research Directions



- Clarify definitions of operators on vector- and matrix-valued cochains.
- Define interpolants $\mathcal{I}_k$ and projections $\mathcal{P}_k$ for these spaces.
- Derive model conformity criteria for the elasticity complex.

Questions?

- Thanks for inviting me to speak!
- Slides available at http://www.ma.utexas.edu/users/agillette