$H(\text{curl})$ and $H(\text{div})$ Elements on Polytopes from Generalized Barycentric Coordinates

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joint work with

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Chandrajit Bajaj (UT Austin)
The generalized barycentric coordinate approach

Let $P$ be a convex polytope with vertex set $V$. We say that $\lambda_v : P \to \mathbb{R}$ are \textbf{generalized barycentric coordinates (GBCs)} on $P$ if they satisfy $\lambda_v \geq 0$ on $P$ and $L = \sum_{v \in V} L(v_v)\lambda_v$, $\forall L : P \to \mathbb{R}$ linear.

Familiar properties are implied by this definition:

\[
\sum_{v \in V} \lambda_v \equiv 1 \quad \text{partition of unity} \\
\sum_{v \in V} v\lambda_v(x) = x \quad \text{linear precision} \\
\lambda_{v_i}(v_j) = \delta_{ij} \quad \text{interpolation}
\]

traditional FEM

family of \textbf{GBC} reference elements
Developments in GBC FEM theory

1. Characterization of the dependence of error estimates on polytope geometry.

2. Construction of higher order scalar-valued methods using $\lambda_v$ functions.

3. Construction of $H(\text{curl})$ and $H(\text{div})$ methods using $\lambda_v$ and $\nabla \lambda_v$ functions.

$H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$

$\{\lambda_i\} \quad \{\lambda_i \nabla \lambda_j\} \quad \{\lambda_i \nabla \lambda_j \times \nabla \lambda_k\} \quad \{\chi_P\}$
Many choices of generalized barycentric coordinates

- **Triangulation**
  \[ \Rightarrow \text{FLOATER, HORMANN, Kós, A general construction of barycentric coordinates over convex polygons, 2006} \]
  \[ 0 \leq \lambda_{T}^{m}(x) \leq \lambda_{i}(x) \leq \lambda_{T}^{M}(x) \leq 1 \]

- **Wachspress**
  \[ \Rightarrow \text{WACHSPRESS, A Rational Finite Element Basis, 1975.} \]
  \[ \Rightarrow \text{WARREN, Barycentric coordinates for convex polytopes, 1996.} \]

- **Sibson / Laplace**
  \[ \Rightarrow \text{SIBSON, A vector identity for the Dirichlet tessellation, 1980.} \]
  \[ \Rightarrow \text{HIYOSHI, SUGIHARA, Voronoi-based interpolation with higher continuity, 2000.} \]
Many choices of generalized barycentric coordinates

- Mean value
  \[ \Rightarrow \text{ FLOATER, Mean value coordinates, 2003.} \]
  \[ \Rightarrow \text{ FLOATER, Kós, Reimers, Mean value coordinates in 3D, 2005.} \]

- Harmonic
  \[ \Rightarrow \text{ Warren, Schaefer, Hirani, Desbrun, Barycentric coordinates for convex sets, 2007.} \]
  \[ \Rightarrow \text{ Christiansen, A construction of spaces of compatible differential forms on cellular complexes, 2008.} \]

Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, surface barycentric coordinates, etc...)
From scalar to vector elements

The classical finite element sequences for a domain $\Omega \subset \mathbb{R}^n$ are written:

$$
\begin{align*}
  n = 2 : & \quad H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xleftarrow{\text{rot}} H(\text{div}) \xrightarrow{\text{div}} L^2 \\
  n = 3 : & \quad H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2
\end{align*}
$$

These correspond to the $L^2$ deRham diagrams from differential topology:

$$
\begin{align*}
  n = 2 : & \quad H^0 \xrightarrow{d_0} H^1 \xleftarrow{\cong} H^1 \xrightarrow{d_1} H^2 \\
  n = 3 : & \quad H^0 \xrightarrow{d_0} H^1 \xrightarrow{d_1} H^2 \xrightarrow{d_2} H^3
\end{align*}
$$

Conforming finite element subspaces of $H^0$ are of two types:

$$
\mathcal{P}_r^k := k\text{-forms with degree } r \text{ polynomial coefficients}
$$

$$
\mathcal{P}_r^-^k := \mathcal{P}_r^k \oplus \text{certain additional } k\text{-forms}
$$

This notation, from Finite Element Exterior Calculus, can be used to describe many well-known finite element spaces.

### Classical finite element spaces on simplices

#### \( n=2 \) (triangles)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \text{dim} )</th>
<th>space</th>
<th>type</th>
<th>classical description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>( \mathcal{P}_1 \Lambda^0 )</td>
<td>( H^1 )</td>
<td>Lagrange elements of degree ( \leq 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{P}_1 \Lambda^0 )</td>
<td>( H^1 )</td>
<td>Lagrange elements of degree ( \leq 1 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>( \mathcal{P}_1 \Lambda^1 )</td>
<td>( H(\text{div}) )</td>
<td>Brezzi-Douglas-Marini ( H(\text{div}) ) elements of degree ( \leq 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{P}_1 \Lambda^1 )</td>
<td>( H(\text{div}) )</td>
<td>Raviart-Thomas elements of order 0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( \mathcal{P}_1 \Lambda^2 )</td>
<td>( L^2 )</td>
<td>discontinuous linear</td>
</tr>
<tr>
<td>1</td>
<td>( \mathcal{P}_1 \Lambda^2 )</td>
<td>( L^2 )</td>
<td>discontinuous piecewise constant</td>
<td></td>
</tr>
</tbody>
</table>

#### \( n=3 \) (tetrahedra)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \text{dim} )</th>
<th>space</th>
<th>type</th>
<th>classical description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>( \mathcal{P}_1 \Lambda^0 )</td>
<td>( H^1 )</td>
<td>Lagrange elements of degree ( \leq 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( \mathcal{P}_1 \Lambda^0 )</td>
<td>( H^1 )</td>
<td>Lagrange elements of degree ( \leq 1 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>( \mathcal{P}_1 \Lambda^1 )</td>
<td>( H(\text{curl}) )</td>
<td>Nédélec second kind ( H(\text{curl}) ) elements of degree ( \leq 1 )</td>
</tr>
<tr>
<td>6</td>
<td>( \mathcal{P}_1 \Lambda^1 )</td>
<td>( H(\text{curl}) )</td>
<td>Nédélec first kind ( H(\text{curl}) ) elements of order 0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>( \mathcal{P}_1 \Lambda^2 )</td>
<td>( H(\text{div}) )</td>
<td>Nédélec second kind ( H(\text{div}) ) elements of degree ( \leq 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( \mathcal{P}_1 \Lambda^2 )</td>
<td>( H(\text{div}) )</td>
<td>Nédélec first kind ( H(\text{div}) ) elements of order 0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>( \mathcal{P}_1 \Lambda^3 )</td>
<td>( L^2 )</td>
<td>discontinuous linear</td>
</tr>
<tr>
<td>1</td>
<td>( \mathcal{P}_1 \Lambda^3 )</td>
<td>( L^2 )</td>
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<td></td>
</tr>
</tbody>
</table>
## Basis functions on simplices

### $n=2$ (triangles)

<table>
<thead>
<tr>
<th>$k$</th>
<th>dim</th>
<th>space</th>
<th>type</th>
<th>basis functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>$\mathcal{P}_1 \Lambda^0$</td>
<td>$H^1$</td>
<td>$\lambda_i$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$\mathcal{P}_1 \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$\mathcal{P}_1 \Lambda^1$</td>
<td>$H(\text{div})$</td>
<td>$\text{rot}(\lambda_i \nabla \lambda_j)$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$\mathcal{P}_1^\perp \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$\mathcal{P}_1^\perp \Lambda^1$</td>
<td>$H(\text{div})$</td>
<td>$\text{rot}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$\mathcal{P}_1 \Lambda^2$</td>
<td>$L^2$</td>
<td>piecewise linear functions</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\mathcal{P}_1^\perp \Lambda^2$</td>
<td>$L^2$</td>
<td>piecewise constant functions</td>
</tr>
</tbody>
</table>

### $n=3$ (tetrahedra)

<table>
<thead>
<tr>
<th>$k$</th>
<th>dim</th>
<th>space</th>
<th>type</th>
<th>basis functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>$\mathcal{P}_1 \Lambda^0$</td>
<td>$H^1$</td>
<td>$\lambda_i$</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>$\mathcal{P}_1 \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$\mathcal{P}_1 \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>$\mathcal{P}_1 \Lambda^2$</td>
<td>$H(\text{div})$</td>
<td>$\lambda_i \nabla \lambda_j \times \nabla \lambda_k$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>$\mathcal{P}_1 \Lambda^2$</td>
<td>$H(\text{div})$</td>
<td>$(\lambda_i \nabla \lambda_j \times \nabla \lambda_k) + (\lambda_j \nabla \lambda_k \times \nabla \lambda_i) + (\lambda_k \nabla \lambda_i \times \nabla \lambda_j)$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$\mathcal{P}_1 \Lambda^3$</td>
<td>$L^2$</td>
<td>piecewise linear functions</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\mathcal{P}_1^\perp \Lambda^3$</td>
<td>$L^2$</td>
<td>piecewise constant functions</td>
</tr>
</tbody>
</table>
The vector-valued basis constructions \((0 < k < n)\) have two key properties:

1. **Global continuity in** \(H(\text{curl})\) or \(H(\text{div})\)

   - \(\lambda_i \nabla \lambda_j\) agree on **tangential** components at element interfaces \(\implies H(\text{curl})\) continuity
   - \(\lambda_i \nabla \lambda_j \times \nabla \lambda_k\) agree on **normal** components at element interfaces \(\implies H(\text{div})\) continuity

2. **Reproduction of requisite polynomial differential forms.**

   For \(i, j \in \{1, 2, 3\}\):
   
   \[
   \text{span}\{\lambda_i \nabla \lambda_j\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right\} \cong \mathcal{P}_1 \Lambda^1(\mathbb{R}^2)
   \]
   
   \[
   \text{span}\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\} \cong \mathcal{P}_1^- \Lambda^1(\mathbb{R}^2)
   \]

Using generalized barycentric coordinates, we can extend all these results to polygonal and polyhedral elements.
Basis functions on polygons and polyhedra

**Theorem [G., Rand, Bajaj, 2014]**

Let $P$ be a convex polygon or polyhedron. Given any set of generalized barycentric coordinates $\{\lambda_i\}$ associated to $P$, the functions listed below have *global continuity* and polynomial differential form *reproduction* properties as indicated.

<table>
<thead>
<tr>
<th>$k$</th>
<th>space</th>
<th>type</th>
<th>functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=2$ (polygons)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$P_1 \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j$</td>
</tr>
<tr>
<td></td>
<td>$P_1 \Lambda^1$</td>
<td>$H(\text{div})$</td>
<td>$\text{rot}(\lambda_i \nabla \lambda_j)$</td>
</tr>
<tr>
<td></td>
<td>$P_{-1} \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$</td>
</tr>
<tr>
<td></td>
<td>$P_{-1} \Lambda^1$</td>
<td>$H(\text{div})$</td>
<td>$\text{rot}(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$P_1 \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j$</td>
</tr>
<tr>
<td></td>
<td>$P_{-1} \Lambda^1$</td>
<td>$H(\text{curl})$</td>
<td>$\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$</td>
</tr>
<tr>
<td>$n=3$ (polyhedra)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$P_1 \Lambda^2$</td>
<td>$H(\text{div})$</td>
<td>$\lambda_i \nabla \lambda_j \times \nabla \lambda_k$</td>
</tr>
<tr>
<td></td>
<td>$P_{-1} \Lambda^2$</td>
<td>$H(\text{div})$</td>
<td>$(\lambda_i \nabla \lambda_j \times \nabla \lambda_k) + (\lambda_j \nabla \lambda_k \times \nabla \lambda_i)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+$ $(\lambda_k \nabla \lambda_i \times \nabla \lambda_j)$</td>
</tr>
</tbody>
</table>

**Note:** The indices range over *all* pairs or triples of vertex indices from $P$. 

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Vector Elements on Polytopes with GBCs  
Finite Element Circus 2014  
10 / 13
Let $P \subset \mathbb{R}^3$ be a convex polyhedron with vertex set $\{v_i\}$. Let $x = [x \ y \ z]^T$.

Then for any $3 \times 3$ real matrix $A$,

$$
\begin{align*}
\sum_{i,j} \lambda_i \nabla \lambda_j (v_j - v_i)^T &= \mathbb{I} \\
\sum_{i,j} (A v_i \cdot v_j)(\lambda_i \nabla \lambda_j) &= A x
\end{align*}
$$

$$
\frac{1}{2} \sum_{i,j,k} \lambda_i \nabla \lambda_j \times \nabla \lambda_k ((v_j - v_i) \times (v_k - v_i))^T = \mathbb{I}
$$

$$
\frac{1}{2} \sum_{i,j,k} (A v_i \cdot (v_j \times v_k))(\lambda_i \nabla \lambda_j \times \nabla \lambda_k) = A x.
$$

By appropriate choice of constant entries for $A$, the column vectors of $\mathbb{I}$ and $A x$ span $P_1 \Lambda^1 \subset H(\text{curl})$ or $P_1 \Lambda^2 \subset H(\text{div})$.

$\rightarrow$ Additional identities for the remaining cases are stated in:

Reducing the basis

In some cases, it should be possible to reduce the size of the basis constructed by our method, in an analogous fashion to the quadratic scalar case.

$n=2$ (polygons)

<table>
<thead>
<tr>
<th>k</th>
<th>space</th>
<th># construction</th>
<th># boundary</th>
<th># polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathcal{P}_1 \Lambda^0(m)/\mathcal{P}_0^- \Lambda^1(m)$</td>
<td>$\nu$</td>
<td>$\nu$</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>$\mathcal{P}_1 \Lambda^1(m)$</td>
<td>$\nu(\nu - 1)$</td>
<td>$2e$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{P}_1^- \Lambda^1(m)$</td>
<td>$(\begin{pmatrix} \nu \ 2 \end{pmatrix})$</td>
<td>$e$</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>$\mathcal{P}_1 \Lambda^2(m)$</td>
<td>$\frac{\nu(\nu - 1)(\nu - 2)}{2}$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{P}_1^- \Lambda^2(m)$</td>
<td>$(\begin{pmatrix} \nu \ 3 \end{pmatrix})$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

→ The $n = 3$ (polyhedra) version of this table is given in:

Acknowledgments

Chandrjit Bajaj       UT Austin
Alexander Rand       UT Austin / CD-adapco

Happy birthday, Doug!

Slides and pre-prints:  http://math.arizona.edu/~agillette/
More on GBCs:           http://www.inf.usi.ch/hormann/barycentric