EXTERIOR CALCULUS FOR GENERALIZED BARYCENTRIC INTERPOLATION

National Biomedical Computation Resource

Andrew Gillette[1], Alexander Rand[2], Chandrajit Bajaj[2]

1. Department of Mathematics, UC San Diego

2. Institute for Computational Engineering and Sciences, UT Austin

Computational Visualization Center

1. Abstract

Generalized barycentric coordinate functions $\{\lambda_i\}$ allow the extension of finite element exterior calculus (FEEC) from meshes of simplices or cubes to meshes of convex polygons and polyhedra with only mild assumptions on element geometry. We show that the nodal interpolation operator I_ℓ on polygons has an $O(h)$ *a priori* error bound, even allowing for hanging nodes in some cases. We also construct a Lagrange-like operator I_q on polygons with an $O(h^2)$ *a priori* error bound whose degrees of freedom are associated to vertices and edge midpoints. On a square mesh, our construction recovers the classical serendipity space $\mathcal{S}_2(I^2)$ [1]. It has been shown that mapped $\mathcal{S}_2(I^2)$ elements converge only linearly when mapped non-affinely [2], whereas our approach maintains quadratic convergence rates even on meshes of non-affine maps of squares.

2. Generalized Barycentric Coordinate Types

Definition. Let Ω be a convex polygon or polyhedron with vertices $\mathbf{v}_i$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are **barycentric coordinates** on Ω if they satisfy two properties.

B1. **Non-negative**: $\lambda_i \geq 0$ on Ω.

B2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $\quad L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$,

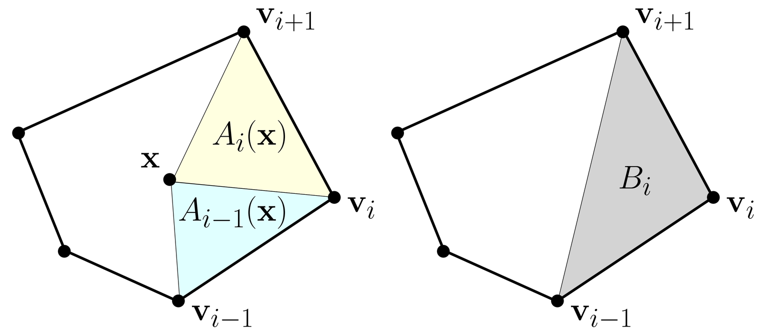
Remark: B2 implies span$\{\lambda_i\} \supset \mathcal{P}_1\Lambda^0(\Omega)$ = linear polynomials on Ω.

We restrict to coordinates that are invariant under rotation, translation, and uniform scaling.

Wachspress Coordinates [8]

$$w_i^{\text{Wach}}(\mathbf{x}) := B_i \prod_{j \neq i, i-1} A_j(\mathbf{x}).$$

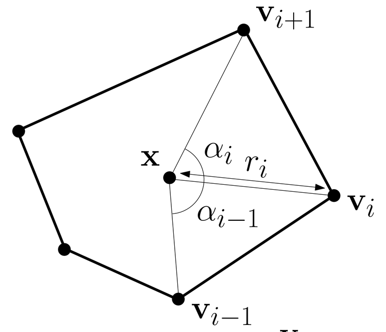
$$\lambda_i^{\text{Wach}}(\mathbf{x}) := \frac{w_i^{\text{Wach}}(\mathbf{x})}{\sum_{j=1}^{n} w_j^{\text{Wach}}(\mathbf{x})}$$



Mean Value Coordinates [4]

$$w_i^{\text{MVal}}(\mathbf{x}) := \frac{\tan(\alpha_i(\mathbf{x})/2) + \tan(\alpha_{i-1}(\mathbf{x})/2)}{||\mathbf{v}_i - \mathbf{x}||}$$

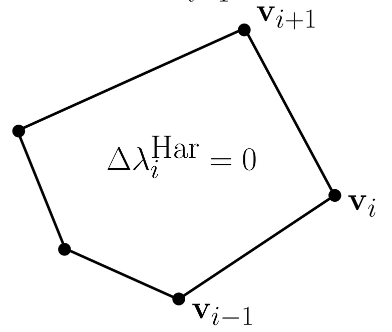
$$\lambda_i^{\text{MVal}}(\mathbf{x}) = \frac{w_i^{\text{MVal}}(\mathbf{x})}{\sum_{j=1}^{n} w_j^{\text{MVal}}(\mathbf{x})}.$$



Harmonic Coordinates

Solve $\begin{cases} \Delta\left(\lambda_i^{\text{Har}}\right) = 0, & \text{on } \Omega, \\ \lambda_i^{\text{Har}} = g_i. & \text{on } \partial\Omega. \end{cases}$

where $g_i(\mathbf{v}_j) = \delta_{ij}$ is piecewise linear.



Other types: **Triangulation** (λ^{Tri}), **Sibson** (λ^{Sibs}), and more...

3. Interpolation Properties

Define a **linear** interpolation operator $I_\ell : H^2(\Omega) \to$ span$\{\lambda_i\}$:

$$I_\ell u := \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \qquad (1)$$

Theorem: [5,6] For any convex polygon Ω satisfying mild geometrical bounds, λ^{Wach}, λ^{MVal}, λ^{Har}, λ^{Tri}, and λ^{Sibs} are bounded in H^1, i.e. $\exists\, C > 0$ such that $||\lambda_i||_{H^1(\Omega)} \leq C$.

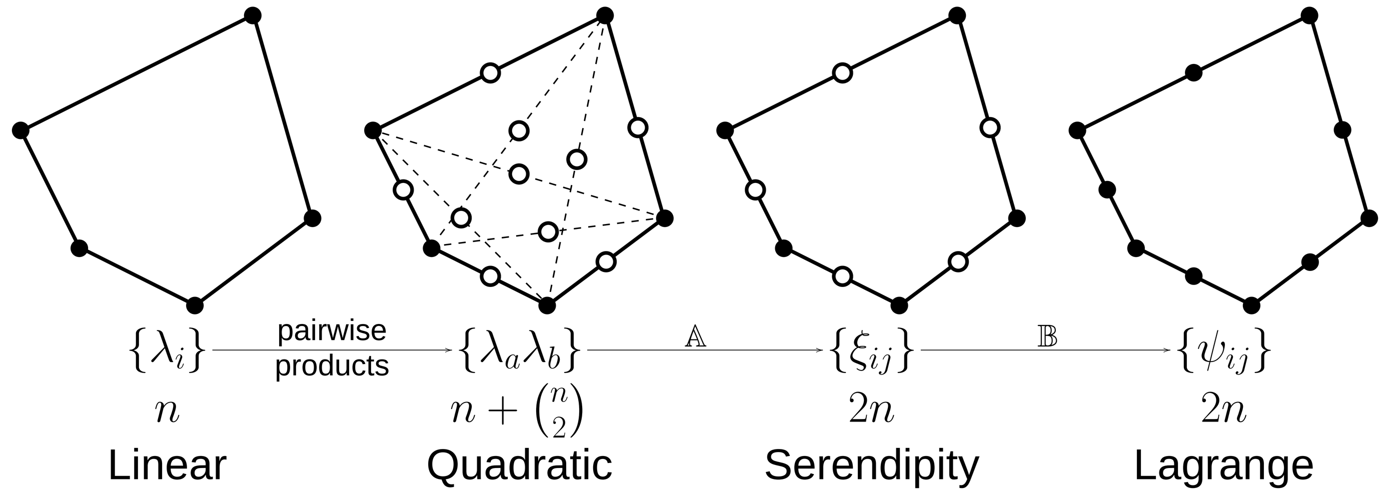
This provides an optimal (linear) *a priori* error estimate:

$$||u - I_\ell u||_{H^1(\Omega)} \leq c\, \text{diam}(\Omega)\, |u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega)$$

Define a **quadratic serendipity** interpolation operator $I_q : H^3(\Omega) \to$ span$\{\lambda_a\lambda_b\}$:

$$I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)} \qquad (2)$$

Here, $[\psi_{ij}]_{j\in\{i,i+1\}} := \mathbb{BA}\,[\lambda_a\lambda_b]$, where the matrix $\mathbb{A}$ reduces the basis size to $2n$:



Theorem: [7] For any convex polygon Ω satisfying mild geometrical bounds and any set of λ_i, we can construct matrices $\mathbb{A}$ and $\mathbb{B}$ such that

i. The entries depend only on the coordinates of the vertices $\mathbf{v}_i$

ii. span$\{\psi_{ij}\} \supset \mathcal{P}_2\Lambda^0(\mathbb{R}^2)$ = quadratic polynomials in x and y

iii. $||\mathbb{A}||$, $||\mathbb{B}||$ bounded in maximum absolute row sum norm.

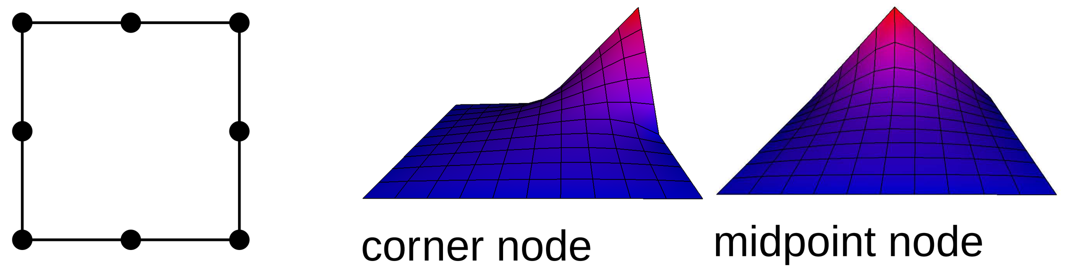
This provides an optimal (quadratic) *a priori* error estimate:

$$||u - I_q u||_{H^1(\Omega)} \leq c\, \text{diam}(\Omega)^2\, |u|_{H^3(\Omega)}, \quad \forall u \in H^3(\Omega)$$

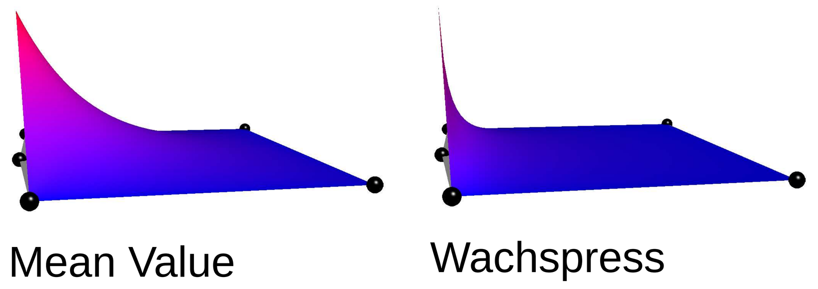
Remark: On $[0,1]^2$, span$\{\psi_{ij}\} \supset \mathcal{S}_2(I^2) = \mathcal{P}_2 \cup \{xy^2, x^2y\}$, as expected (cf. [1])

4. Numerical Examples and Applications

The Mean value coordinates $\{\lambda_i^{\text{MVal}}\}$ are bounded in H^1, on meshes with hanging nodes...

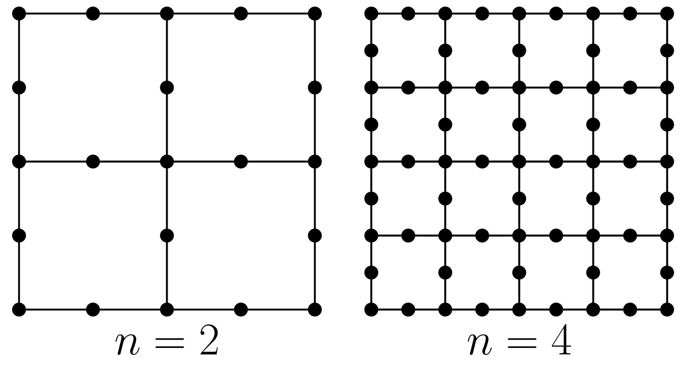


... but the Wachspress coordinates $\{\lambda_i^{\text{Wach}}\}$ have an unbounded gradient, as interior angles get large.



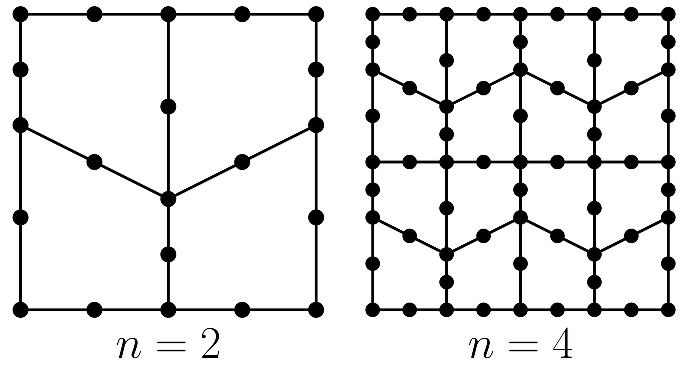
For the Poisson problem $\Delta u = f$, Dirichlet boundary conditions, with exact solution $u(x,y) = \sin(x)e^y$, we observe the expected rates.

Using **linear** interpolation $u_h := I_\ell u$ with λ^{MVal}



	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h\|\|_{L^2}$	
n	error	rate	error	rate
2	3.35e-3		7.56e-2	
4	8.67e-4	2.03	3.60e-2	1.07
8	2.18e-4	1.99	1.76e-2	1.03
16	5.50e-5	1.99	8.73e-3	1.01
32	1.38e-5	1.99	4.35e-3	1.00
64	3.47e-6	1.99	2.17e-3	1.00
128	8.69e-7	2.00	1.09e-3	1.00

Using **quadratic serendipity** interpolation $u_h := I_q u$ with λ^{MVal}



	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

Compare this last case to the non-affine mapping of $\mathcal{S}_2(I^2)$ elements. By $n = 64$, the error rate of $||u - u_h||_{L^2}$ is only 2.1 and the error rate of $||\nabla(u - u_h)||_{L^2}$ is only 1.1 (cf. [2]).

5. Discussion and Future Work

The methods analyzed here suggest a variety of research directions, both theoretical and computational. Theoretical issues include the development of vector elements for interpolation in $H(\text{curl})$ and $H(\text{div})$ on polygons as well as an extension of the theory to polyhedral 3D meshes. Computational issues include the implementation of fast and robust solvers as well as the possible incorporation of these types of flexible elements into existing finite element codes. These techniques give insight toward a higher order version of Discrete Exterior Calculus [3] as well as FEEC theory on a more general class of mesh types.

References

[1] D. N. Arnold and G. Awanou. The serendipity family of finite elements. *Foundations of Computational Mathematics*, 11(3):337–344, 2011.

[2] D. N. Arnold, D. Boffi, and R. S. Falk. Approximation by quadrilateral finite elements. *Math. Comput.*, 71(239):909–922, 2002.

[3] M. Desbrun, A. N. Hirani, M. Leok, and J. E. Marsden. Discrete Exterior Calculus. *arXiv:math/0508341*, 2005.

[4] M. Floater, K. Hormann, and G. Kós. A general construction of barycentric coordinates over convex polygons. *Adv. Comput. Math.*, 24(1):311–331, 2006.

[5] A. Gillette, A. Rand, and C. Bajaj. Error estimates for generalized barycentric interpolation. *Adv. Comput. Math.*, in press, 2011.

[6] A. Rand, A. Gillette, and C. Bajaj. Interpolation error estimates for mean value coordinates. *arXiv:1111.5588*, submitted, 2011.

[7] A. Rand, A. Gillette, and C. Bajaj. Quadratic serendipity finite elements on polygons using generalized barycentric coordinates. *arXiv:1109.3259*, submitted, 2011.

[8] E. L. Wachspress. *A Rational Finite Element Basis*, volume 114 of *Mathematics in Science and Engineering*. Academic Press, 1975.

Institute for Computational and Experimental Research in Mathematics

NSF/CBMS Conference: Finite Element Exterior Calculus (June 2012)