Nodal Basis Functions for Serendipity Finite Elements

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joint work with Michael Floater (University of Oslo)
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Outline

1. Introduction and Motivation
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What is a serendipity finite element method?

Goal: Efficient, accurate approximation of the solution to a PDE over $\Omega \subset \mathbb{R}^n$. Standard $O(h^r)$ tensor product finite element method in $\mathbb{R}^n$:

→ Mesh $\Omega$ by $n$-dimensional cubes of side length $h$.
→ Set up a linear system involving $(r + 1)^n$ degrees of freedom (DoFs) per cube.
→ For unknown continuous solution $u$ and computed discrete approximation $u_h$:

$$||u - u_h||_{H^1(\Omega)} \leq C h^r |u|_{H^{r+1}(\Omega)}, \quad \forall u \in H^{r+1}(\Omega).$$

A $O(h^r)$ serendipity FEM converges at the same rate with fewer DoFs per element:

<table>
<thead>
<tr>
<th>tensor product elements</th>
<th>$O(h)$</th>
<th>$O(h^2)$</th>
<th>$O(h^3)$</th>
<th>$O(h)$</th>
<th>$O(h^2)$</th>
<th>$O(h^3)$</th>
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<tr>
<td>serendipity elements</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
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Example: For $O(h^3)$, $n = 3$, 50% fewer DoFs $\rightarrow \approx 50\%$ smaller linear system
Motivations and Related Topics

Serendipity elements are an essential tool in modern efforts to robustly implement and accelerate high order computational methods.

- **Isogeometric analysis:** Finding basis functions suitable for both domain description and PDE approximation avoids the expensive computational bottleneck of re-meshing.
  

- **Modern mathematics:** Finite Element Exterior Calculus, Discrete Exterior Calculus, Virtual Element Methods...
  
  

- **Flexible Domain Meshing:** Serendipity type elements for Voronoi meshes provide computational benefits without need of tensor product structure.
  
Mathematical challenges

→ Basis functions must be constructed to implement serendipity elements.
→ Current constructions lack key mathematical properties, limiting their broader usage

**Goal:** Construct basis functions for serendipity elements satisfying the following:

- **Symmetry:** Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.
- **Tensor product structure:** Write as linear combinations of standard tensor product functions.
- **Dimensional nesting:** Generalize to methods on $n$-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.

![Graphs showing the growth of degrees of freedom for different $n$ and $h$ values](image-url)
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Which monomials do we need?

\[ O(h^3) \]
serendipity element:

\[ \{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, x^3y, xy^3, x^2y^2, x^3y^2, x^3y^3, x^2y^3, x^3y^3\} \]

- total degree at most cubic
  (req. for \( O(h^3) \) approximation)

- at most cubic in each variable
  (used in \( O(h^3) \) tensor product methods)

We need an intermediate set of 12 monomials!

The superlinear degree of a polynomial ignores linearly-appearing variables.

**Example:** \( \text{sldeg}(xy^3) = 3 \), even though \( \text{deg}(xy^3) = 4 \)

**Definition:** \( \text{sldeg}(x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}) := \left( \sum_{i=1}^{n} e_i \right) - \# \{e_i : e_i = 1\} \)

\[ \{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, x^3y, xy^3, x^2y^2, x^3y^2, x^3y^3\} \]

superlinear degree at most 3 (dim=12)

Superlinear polynomials form a lower set

Given a monomial

\[ x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \]

associate the multi-index of \( n \) non-negative integers

\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n. \]

Define the superlinear norm of \( \alpha \) as

\[ |\alpha|_{sprlin} := \sum_{\substack{j=1 \\ \alpha_j \geq 2}}^{n} \alpha_j, \]

so that the superlinear multi indices are

\[ S_r = \{ \alpha \in \mathbb{N}_0^n : |\alpha|_{sprlin} \leq r \}. \]

Observe that \( S_r \) has a partial ordering

\[ \mu \leq \alpha \text{ means } \mu_i \leq \alpha_i. \]

Thus \( S_r \) is a lower set, meaning

\[ \alpha \in S_r, \mu \leq \alpha \implies \mu \in S_r \]

Theorem (Dyn and Floater, 2013)

Fix a lower set \( L \subset \mathbb{N}_0^n \) and points \( z_\alpha \in \mathbb{R}^n \) for all \( \alpha \in L \). For any sufficiently smooth \( n \)-variate real function \( f \), there is a unique polynomial \( p \) in \( \text{span}\{x^\alpha : \alpha \in L \} \) that interpolates \( f \) at the points \( z_\alpha \), with partial derivative interpolation for repeated \( z_\alpha \).

Dyn and Floater Multivariate polynomial interpolation on lower sets, J. Approx. Th., to appear.
Partitioning and reordering the multi-indices

By a judicious choice of the interpolation points \( z_\alpha = (x_i, y_j) \), we recover the dimensionality associations of the degrees of freedom of serendipity elements.

The order 5 serendipity element, with degrees of freedom color-coded by dimensionality.

The lower set \( S_5 \), with equivalent color coding.

The lower set \( S_5 \), with domain points \( z_\alpha \) reordered.
Symmetrizing the multi-indices

By collecting the re-ordered interpolation points \( z_\alpha = (x_i, y_j) \), at midpoints of the associated face, we recover the dimensionality associations of the degrees of freedom of serendipity elements.

The lower set \( S_5 \), with domain points \( z_\alpha \) reordered.

A symmetric reordering, with multiplicity. The associated interpolant recovers values at dots, three partial derivatives at edge midpoints, and two partial derivatives at the face midpoint.
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Symmetry: Accommodate interior degrees of freedom that grow according to triangular numbers on square-shaped elements.
The Dyn-Floater interpolation scheme is expressed in terms of tensor product interpolation over ‘maximal blocks’ in the set using an inclusion-exclusion formula.

Put differently, the linear combination is the sum over all blocks within the lower set with coefficients determined as follows:

→ Place the coefficient calculator at the extremal block corner.
→ Add up all values appearing in the lower set.
→ The coefficient for the block is the value of the sum.

Hence: black dots → +1; white dots → -1; others → 0.
Thus, using our symmetric approach, each maximal block in the lower set becomes a standard tensor-product interpolant.
Tensor product structure: Write basis functions as linear combinations of standard tensor product functions.
**3D elements**

**Dimensional nesting:** Generalize to methods on $n$-cubes for any $n \geq 2$, allowing restrictions to lower-dimensional faces.
3D coefficient computation

Lower sets for superlinear polynomials in 3 variables:

Decomposition into a linear combination of tensor product interpolants works the same as in 2D, using the 3D coefficient calculator at left. (blue $\rightarrow$ +1; orange $\rightarrow$ -1).

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Future Directions

- Incorporate elements into finite element software packages.
- Analyze speed vs. accuracy trade-offs.

<table>
<thead>
<tr>
<th>$n = 2$</th>
<th>$n = 3$</th>
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<tbody>
<tr>
<td>$\dim Q_r$</td>
<td>$\dim Q_r$</td>
</tr>
<tr>
<td>$\dim S_r$</td>
<td>$\dim S_r$</td>
</tr>
<tr>
<td>$1$</td>
<td>$8$</td>
</tr>
<tr>
<td>$2$</td>
<td>$9$</td>
</tr>
<tr>
<td>$3$</td>
<td>$16$</td>
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<td>$4$</td>
<td>$25$</td>
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<td>$5$</td>
<td>$36$</td>
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<tr>
<td>$6$</td>
<td>$49$</td>
</tr>
<tr>
<td>$7$</td>
<td>$64$</td>
</tr>
<tr>
<td>$r \geq 2n$</td>
<td>$r \geq 3n$</td>
</tr>
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$r^2 + 2r + 1$ \hspace{1cm} $\frac{1}{2}(r^2 + 3r + 6)$

Expand serendipity results to generic polygons and polyhedra.
Acknowledgments

Michael Holst
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Michael Floater
University of Oslo

Thanks for your attention!

Slides and pre-prints:  http://math.arizona.edu/~agillette/