1. The form of Simpson’s Rule given in the book is

\[ \text{SIMP}(n) = \frac{1}{3} [2\text{MID}(n) + \text{TRAP}(n)] \]

But

\[ \text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2} \]

So SIMP(n) is really

\[ \frac{1}{3} [\left( \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2} \right) + 2\text{MID}(n)] \]

or

\[ \frac{1}{3} (\text{LEFT}(n)/2 + \text{RIGHT}(n)/2 + 2\text{MID}(n)) \]

2. So this suggests a refinement of the Riemann sum that approximates

\[ \int_{a}^{b} f(x)dx. \]

We will want to use three points in each subinterval of the partition for the Riemann Sum, the left and right-hand endpoints, and the midpoint. To use all three points we will fit a parabola through them. So the first step is to find the area under a piece of a parabola in terms of these three points. We shall assault this problem in steps.

3. The book claims that if \( f(x) \) is quadratic, then this formula applies:

\[ \int_{a}^{b} f(x)dx = \frac{h}{3} \left( \frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2} \right) \]

where \( h = b - a \) and the midpoint is \( m = (b - a)/2 \). Let us verify this, and then apply it to the parabola approximates the function over one strip in the Riemann Sum.

4. A general parabola looks like \( Ax^2 + Bx + C \) and we need to get

\[ \int_{a}^{b} Ax^2 + Bx + C \, dx = A \int_{a}^{b} x^2 \, dx + B \int_{a}^{b} x \, dx + C \int_{a}^{b} 1 \, dx \]
a. If \( f(x) = 1 \) then

\[ \int_a^b f(x)\,dx = (b - a) \]

and

\[ \frac{h}{3} \left( \frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2} \right) = \frac{(b - a)}{3} \left( \frac{1}{2} + 2 + \frac{1}{2} \right) \]

which is \((b - a)\), so the formula works for \( f(x) = 1 \).

b. We do the same for \( f(x) = x \).

\[ \int_a^b x\,dx = \frac{b^2 - a^2}{2} \]

and

\[ \frac{h}{3} \left( \frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2} \right) = \frac{b - a}{3} \left( \frac{a}{2} + \frac{2a + b}{2} + \frac{b}{2} \right) \]
\[ = \frac{b - a}{3} \left( \frac{3a + 3b}{2} \right) \]
\[ = \frac{b - a}{3} \left( b + a \right) \]
\[ = \frac{b^2 - a^2}{2} \]

and so the formula works for \( f(x) = x \).

c. For \( f(x) = x^2 \),

\[ \int_a^b f(x)\,dx = \frac{b^3 - a^3}{3} \]

and

\[ \frac{h}{3} \left( \frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2} \right) = \frac{b - a}{3} \left( \frac{a^2}{2} + 2 \left( \frac{a + b}{2} \right)^2 + \frac{b^2}{2} \right) \]
\[
\begin{align*}
&= \frac{b - a}{3} \left( \frac{a^2}{2} + \frac{a^2 + 2ab + b^2}{2} + \frac{b^2}{2} \right) \\
&= \frac{b - a}{3} \left( \frac{2a^2 + 2ab + 2b^2}{2} \right) \\
&= \frac{b - a}{3} (a^2 + ab + b^2) \\
&= \frac{b^3 - a^3}{3}
\end{align*}
\]

and again it works.

5. For our general quadratic, \( f(x) = Ax^2 + Bx + C \), we have

\[
\int_a^b Ax^2 + Bx + C \, dx = A \int_a^b x^2 \, dx + B \int_a^b x \, dx + C \int_a^b 1 \, dx
\]

and using the above results we get

\[
\int_a^b f(x) \, dx = A \frac{h}{3} \left( \frac{a^2}{2} + 2m^2 + \frac{b^2}{2} \right) + B \frac{h}{3} \left( \frac{a}{2} + 2m + \frac{b}{2} \right) + C \frac{h}{3}(3)
\]

and we get the same formula for the general quadratic that we got for the pieces.

6. Now we are home free. To get the Riemann sum for

\[
\int_a^b f(x) \, dx
\]

using the quadratic approximation, assuming \( f(x) \) is any integrable function, we partition the interval \([a, b]\), and let \( q_i(x) \) be the quadratic approximation to \( f(x) \) on the subinterval \([x_i, x_{i+1}]\). Let \( m_i \) be the midpoint of this subinterval, and \( \Delta x = h = (x_{i+1} - x_i) \). Our \( q_i \) passes through the endpoints and the midpoint of the subinterval. Then
\[
\int_{x_i}^{x_{i+1}} f(x) \, dx \approx \int_{x_i}^{x_{i+1}} q_i(x) \, dx
\]

and using the results above on this second approximating integral we get

\[
\frac{\Delta x}{3} \left( \frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2} \right)
\]

Now we sum over all the subintervals

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} q_i(x) \, dx = \sum_{i=0}^{n-1} \frac{\Delta x}{3} \left( \frac{q_i(x_i)}{2} + 2q_i(m_i) + \frac{q_i(x_{i+1})}{2} \right)
\]

Splitting the sum into two parts we get,

\[
\frac{2}{3} \sum_{i=0}^{n-1} q_i(m_i) \Delta x + \frac{1}{3} \sum_{i=0}^{n-1} \left( \frac{q_i(x_i)}{2} + \frac{q_i(x_{i+1})}{2} \right) \Delta x
\]

\[
= \frac{2}{3} \text{MID}(n) + \frac{1}{3} \text{TRAP}(n)
\]

\[
= \frac{1}{3} \left[ 2\text{MID}(n) + \text{TRAP}(n) \right]
\]

\[
= \text{SIMP}(n)
\]

as was to be shown.

7. Who’s afraid of a little algebra??