A Most Amazing Formula

As far as is apparent from casual inspection, the single variable integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx$$

is only remotely connected with circles. Not so, and the calculation of this integral uses some of the tricks we have learned in Math-223, and provides us with a most amazing formula. Let us examine a double indefinite integral which clearly is associated with circles. Let

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy \\ &= \lim_{r\to\infty} \int_{x^2+y^2\leq r^2} e^{-x^2} \cdot e^{-y^2} dxdy \\ &= \lim_{r\to\infty} \int_0^r \int_0^{2\pi} e^{-r^2} \cdot rdrd\theta \\ &= \lim_{b\to\infty} 2\pi \left(-\frac{1}{2} e^{-r^2} \right|_0^b = \pi \end{aligned}$$

Note the shift to polar coordinates.

But we also see that since the variable of integration is irrelevant to the value of an improper integral,

$$\int_{-\infty}^{+\infty} e^{x^2} dx = \int_{-\infty}^{+\infty} e^{y^2} dy$$

and we can write

$$\begin{aligned} I &= \lim_{m\to\infty} \int_{-m}^{m} \int_{-m}^{m} e^{-x^2-y^2} dxdy \\ &= \lim_{m\to\infty} \left(\int_{-m}^{m} e^{-x^2} dx \right) \left(\int_{-m}^{m} e^{-y^2} dy \right) \\ &= \lim_{m\to\infty} \left(\int_{-m}^{m} e^{-x^2} dx \right)^2 \end{aligned}$$

So

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 = \pi$$

And

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Thus the integral of an exponential turns into the square root of the ratio of a circle's circumference to its diameter. Most amazing.