Calculation of the Mean of a Binomial

Suppose we have a binomial with number of trials n, and proability of success p. By definition, the mean is

$$\mu = \sum x_i \cdot P(X = x_i) \text{ over all } i$$

and in the case of the $B(n, p)$ this becomes

$$\sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i}$$

The first term is zero, so we get

$$\sum_{i=1}^{n} i \binom{n}{i} p^i q^{n-i}$$

Now $n! = n(n-1)!$ so writing out the

$$\binom{n}{i}$$

we get

$$\begin{aligned}
E(X) &= \sum_{i=1}^{n} i \left(\frac{n!}{i!(n-i)!} \right) p^i q^{n-i} \\
&= \sum_{i=1}^{n} in \left(\frac{(n-1)!}{i!(n-i)!} \right) p^i q^{n-i} \\
\text{Since } i! &= i(i-1)!, \\
E(X) &= \sum_{i=1}^{n} \left(\frac{n!}{(i-1)!(n-i)!} \right) p^i q^{n-i}
\end{aligned}$$

We now add zero (zero-trick) to the denominator, leaving it unchanged. We choose zero in the form:

$$0 = +1 - 1$$

so

$$n - i = n - i + 1 - 1 = (n-1) + (-i+1) = (n-1) - (i-1)$$

Giving us:

$$E(X) = \sum_{i=1}^{n} \left(\frac{(n-1)!}{(i-1)!((n-i)-(i-1)!)} \right) p^i q^{n-i}$$

Next we take the n outside the Sigma-sign and recognize that the large parentheses above are equivalent to

$$\binom{n-i}{i-1.}$$

Thus

$$E(X) = n\sum_{i=1}^{n}\binom{n-1}{i-1}p^i q^{n-i}$$

Next take a p outside the sigma sign since $p^i = p \cdot p^{i-1}$

$$E(X) = np\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1} q^{n-i}$$

and setting $(n-i) = n-i-1+1 = (n-1)-(i-1)$, we get

$$E(X) = np\sum_{i=1}^{n}\binom{n-1}{i-1}p^{i-1} q^{(n-1)-(i-1)}$$

Now shift the index down by one, so let $l = (i-1)$

$$E(X) = np\sum_{l=0}^{n-1}\binom{n-1}{l}p^{l} q^{(n-1)-l}$$

which is np times the binomial expansion of

$$(p+q)^{n-1}$$

so

$$E(X) = np(p+q)^{n-1} = np.$$

The Variance of the binomial is, by definition

$$\text{VAR}(X) = \sum_{i=1}^{n}\binom{n}{i}(i-np)^2 p^i q^{n-i}$$

and the same tricks applied once to p and once to q yield

$$\text{VAR}(X) = npq$$

so

$$\sigma_X = \sqrt{npq}$$

as given in the book.