Southwestern Center for Arithmetical Algebraic Geometry
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Diophantine Geometry Related to the ABC Conjecture

STUDENT PRESENTATION

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Theorem:

Assuming the ABC, Szpiro, and B&SD conjectures, all semistable elliptic curves of fixed rank have

$$|\text{Ш}| \ll N^{3/4+\varepsilon}.$$

From:

Dorian Goldfeld and Lucien Szpiro, *Bounds for the order of the Tate–Shafarevich group*, Compositio Mathematica **97**, 1995, pp. 71–87.

ASSUMPTIONS

ABC Conjecture
For all $A, B, C \in \mathbf{Z}$ with $\mathrm{GCD}(A, B, C) = 1$ and $A + B = C$,

$$\mathrm{SUP}\left(|A|, |B|, |C|\right) \ll N_0(ABC)^{1+\varepsilon},$$

where N_0 is the conductor of the product ABC:

$$N_0(ABC) = \prod_{p|ABC} p.$$

Szpiro's Conjecture
For all elliptic curves $E/\mathbf{Q}$,

$$|\mathcal{D}| \ll N^{6+\varepsilon},$$

where $\mathcal{D}$ is the minimal discriminant of E, and N is the conductor of E.

Remark
The above conjectures are equivalent. We will make use of both formulations.

ASSUMPTIONS

Birch & Swinnerton-Dyer Conjecture

Let $L(s) \equiv L$-series of E, and $r \equiv \text{RANK}(E)$. Then

$$\operatorname*{ORD}_{s=1} L(s) = r,$$

and

$$\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} = \frac{2^r \cdot |\text{Ш}| \cdot \Omega \cdot R \cdot \prod_{p|N} c_p}{|E(\mathbf{Q})_{\text{TOR}}|^2},$$

where Ω is the real period, R is the regulator, and the c_p are the local periods (Tamagawa numbers).

THEOREM

Assuming the previous three conjectures, we have:

For all semistable elliptic curves *of fixed rank* r,

$$|Ш| \ll N^{3/4+\varepsilon}.$$

Precise statement:

For each rank r and each $\varepsilon > 0$, there exists a constant $C_{r,\varepsilon} > 0$ so that, for all semistable elliptic curves of rank r, we have

$$|Ш| < C_{r,\varepsilon} \cdot N^{3/4+\varepsilon}.$$

OVERVIEW OF PROOF

Rearranging B&SD gives:

$$|\text{Ш}| = \left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \cdot \frac{|E(\mathbf{Q})_{\mathsf{TOR}}|^2}{2^r\,\Omega\,R\,\prod_{p|N} c_p} \ll N^{3/4+\varepsilon}$$

(rank fixed)

Thus, to bound $|\text{Ш}|$, we seek bounds for:

- $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}$, to be shown

- $|E(\mathbf{Q})_{\mathsf{TOR}}|^2 \leq 144$, by Mazur's theorem

- $\frac{1}{\Omega} \ll N^{1/2+\varepsilon}$, to be shown

- $\frac{1}{R} \ll C_r$ (constant depending on r),
  to be shown

- $\frac{1}{\prod_{p|N} c_p} \leq 1$, since each $c_p = \left|\frac{E(\mathbf{Q}_p)}{E_0(\mathbf{Q}_p)}\right| \geq 1$

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Thus, to bound $|\text{Ш}|$, we seek bounds for:

- $\left.\dfrac{L(s)}{(s-1)^r}\right|_{s=1}$
- $|E(\mathbf{Q})_{\text{TOR}}|^2$
- $\dfrac{1}{\Omega}$
- $\dfrac{1}{R}$
- $\dfrac{1}{\prod_{p|N} c_p}$

≤ 144, by Mazur's theorem

≤ 1, since each $c_p = \left| \frac{E(\mathbf{Q}_p)}{E_0(\mathbf{Q}_p)} \right| \geq 1$

$\ll N^{1/4+\varepsilon}$, to be shown

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$\ll C_r$ (constant depending on r),
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$\ll N^{3/4+\varepsilon}$

(rank fixed)

Establishing $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}$

Recall, for $\Re(s) > 3/2$, and semistable E:

$$L(s) = \prod_{p|N} \frac{1}{1 \pm p^{-s}} \prod_{p\nmid N} \frac{1}{1 - a_p\, p^{-s} + p^{1-2s}},$$

where $a_p = (1+p) - \#E_p(\mathbf{F}_p)$.

To bound $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1}$, we proceed indirectly. As usual, set

$$\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s),$$

so that $\Lambda(2-s) = \pm\Lambda(s)$.

We have the bounds:

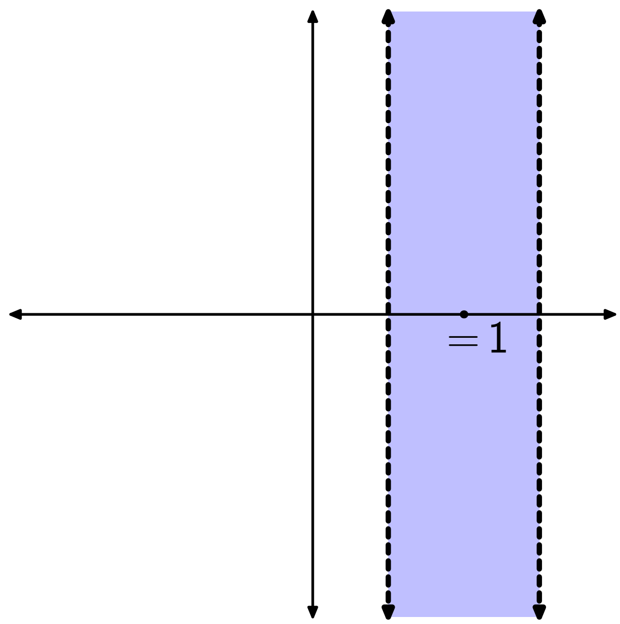
- $\left|\frac{\Lambda(s)}{(s-1)^r}\right| \ll N^{3/4+\varepsilon}$ for $\Re(s) = 3/2+\varepsilon$.

- $\left|\frac{\Lambda(s)}{(s-1)^r}\right| \ll N^{3/4+\varepsilon}$ for $\Re(s) = 1/2-\varepsilon$.

Establishing $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}$

Now we will transfer these bounds to the strip

$$1/2-\varepsilon \le \Re(s) \le 3/2+\varepsilon.$$



By a generalized maximum modulus principle, it is sufficient to show, throughout the strip,

$$\left|\frac{\Lambda(s)}{(s-1)^r}\right| \text{ is bounded (by } \textit{something}\text{).}$$

Establishing $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}$

Obtaining an arbitrary bound on $\left|\frac{\Lambda(s)}{(s-1)^r}\right|$:

- Some upper bound on $\Lambda(s)$:

  $$\Lambda(s) = N^{s/2}\underbrace{(2\pi)^{-s}\Gamma(s)L(s)}_{\substack{\text{Mellin transform of} \\ \text{the weight 2 cusp} \\ \text{form } \sum a_n q^n}} \quad \text{(semistable)}$$

- Some lower bound on $(s-1)^r$:
  B&SD implies that $\frac{\Lambda(s)}{(s-1)^r}$ does not blow up at $s = 1$.

Establishing $\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}$

Thus our bounds transfer to give

$$\left|\frac{\Lambda(s)}{(s-1)^r}\right| \ll N^{3/4} + \varepsilon$$

for $1/2 - \varepsilon \leq \Re(s) \leq 3/2 + \varepsilon$.

In particular, substituting $s = 1$ gives

$$N^{1/2}(2\pi)^{-1}\Gamma(1) \left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{3/4+\varepsilon},$$

so that

$$\left.\frac{L(s)}{(s-1)^r}\right|_{s=1} \ll N^{1/4+\varepsilon}.$$

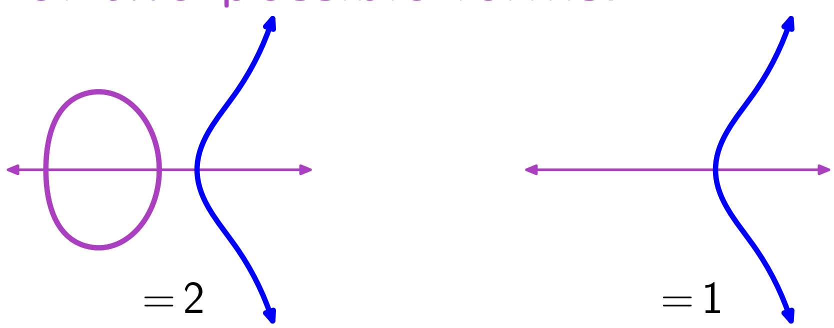
Establishing $\frac{1}{\Omega} \ll N^{1/2+\varepsilon}$

Recall that the real period is

$$\Omega = \int_{E(\mathbf{R})} |\omega| \, ,$$

where ω is the associated invariant differential.

Upon choosing a Weierstrass equation, $E(\mathbf{R})$ has one of two possible forms:



Let γ denote the infinite component, and let δ denote the number of components. The integral along either component is the same; thus

$$\Omega = \delta \int_{\gamma} |\omega| \, .$$

Establishing $\frac{1}{\Omega} \ll N^{1/2+\varepsilon}$

Starting with a global minimal equation

$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6,$$

we have $$\omega = \frac{\mathrm{d}X}{2Y + a_1 X + a_3}.$$

The usual transformations (see Tate's article or Silverman's book) give the convenient form:

$$\begin{aligned} y^2 &= x^3 - 27c_4 x - 54c_6 \\ \omega &= \frac{3\mathrm{d}x}{y}. \end{aligned}$$

Furthermore (we'll need this later):

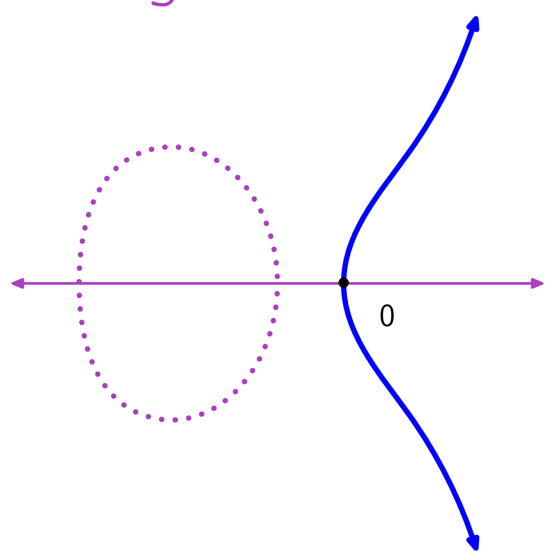
$$\begin{aligned} 1728\mathcal{D} &= {c_4}^3 - {c_6}^2, \\ \mathrm{GCD}(c_4, \mathcal{D}) &= 1. \qquad \leftarrow \text{semistable} \end{aligned}$$

Establishing $\frac{1}{\Omega} \ll N^{1/2+\varepsilon}$

Then $\Omega = \delta \int_\gamma |\omega|$ becomes

$$\Omega = \delta \int_\gamma \left| \frac{3\,\mathrm{d}x}{y} \right| = 2\delta \int_{r_0}^{\infty} \frac{3\,\mathrm{d}x}{\sqrt{x^3 - 27c_4x^2 - 54c_6}},$$

where r_0 is the largest real root of the cubic.



Knowing how to compute Ω, we now show how to bound $1/\Omega$:

$$\frac{1}{\Omega} \underset{\substack{\uparrow \\ \text{Calculus}}}{\ll} \mathrm{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right) \underset{\substack{\uparrow \\ \text{ABC}}}{\ll} N^{1/2+\varepsilon}.$$

Calculus: $\frac{1}{\Omega} \ll \text{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right)$

We must study

$$\Omega = 2\delta \int_{r_0}^{\infty} \frac{3\,\mathrm{d}x}{\sqrt{x^3 - 27c_4x - 54c_6}}.$$

We will consider two cases based on the value

$$k = |c_4| \,/\, |c_6|^{2/3}.$$

If $|c_6|^2 \geq |c_4|^3$, i.e., if $0 < k \leq 1$, then write the integral in the form

$$\frac{1}{|c_6|^{1/2}} \int_{r_0}^{\infty} \frac{3\,\mathrm{d}x}{\sqrt{\left(\frac{x}{|c_6|^{1/3}}\right)^3 \pm 27\left(\frac{|c_4|}{|c_6|^{2/3}}\right)\left(\frac{x}{|c_6|^{1/3}}\right) \pm 54}}.$$

The substitution $u = x/\,|c_6|^{1/3}$ gives

$$\frac{1}{|c_6|^{1/6}} \int_{\underset{\substack{\uparrow \\ \text{largest real root of } u^3 \pm 27ku \pm 54}}{r(k)}}^{\infty} \frac{3\,\mathrm{d}u}{\sqrt{u^3 \pm 27ku \pm 54}}.$$

Calculus: $\frac{1}{\Omega} \ll \text{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right)$

Three observations:

- The integral

$$\frac{1}{|c_6|^{1/6}} \int_{r(k)}^{\infty} \frac{3\,\mathrm{d}u}{\sqrt{u^3 \pm 27ku \pm 54}}$$

  converges when $u^3 \pm 27ku \pm 54$ has distinct roots, i.e., when $k \neq 1$.

- The integral goes to ∞ as $k \to 1$, which is okay since we seek a lower bound.

- The integral is continuous in k for $k \in [0, 1-\varepsilon]$.

Calculus: $\frac{1}{\Omega} \ll \mathrm{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right)$

Hence a lower bound must exist for the integral, so that

$$\frac{1}{|c_6|^{1/6}} \ll \Omega.$$

The other case, $|c_6|^2 \leq |c_4|^3$, similarly gives

$$\frac{1}{|c_4|^{1/4}} \ll \Omega.$$

Combining the two cases gives

$$\frac{1}{\Omega} \ll \mathrm{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right).$$

ABC: $\mathrm{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right) \ll N^{1/2+\varepsilon}$.

Recall

$$\begin{aligned} \mathrm{GCD}(c_4, \mathcal{D}) &= 1, \\ {c_4}^3 - {c_6}^2 &= 1728\mathcal{D}. \end{aligned}$$

Let $d = \mathrm{GCD}({c_4}^3, 1728)$. Applying ABC to

$$\frac{{c_4}^3}{d} - \frac{{c_6}^2}{d} = \frac{1728\mathcal{D}}{d}.$$

gives

$$\begin{aligned} \mathrm{SUP}\left(\frac{|c_4|^3}{d}, \frac{|c_6|^2}{d}\right) &\ll N_0\left(\frac{{c_4}^3{c_6}^2 1728\mathcal{D}}{d^3}\right)^{1+\varepsilon} \\ &\ll N_0\left(c_4 c_6 2 \cdot 3 \cdot \mathcal{D}\right)^{1+\varepsilon} \\ &\ll |c_4|^{1+\varepsilon}\, |c_6|^{1+\varepsilon}\, N_0(\mathcal{D})^{1+\varepsilon} \\ &= |c_4|^{1+\varepsilon}\, |c_6|^{1+\varepsilon}\, N^{1+\varepsilon}. \end{aligned}$$

Absorb the d's into the "$\ll$" constant:

$$\mathrm{SUP}\left(|c_4|^3, |c_6|^2\right) \ll |c_4|^{1+\varepsilon}\, |c_6|^{1+\varepsilon}\, N^{1+\varepsilon}.$$

ABC: $\text{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right) \ll N^{1/2+\varepsilon}$.

There are two cases to consider.
If $|c_6|^2 \le |c_4|^3$, then our inequality

$$\text{SUP}\left(|c_4|^3, |c_6|^2\right) \ll |c_4|^{1+\varepsilon} |c_6|^{1+\varepsilon} N^{1+\varepsilon}$$

becomes $|c_4|^3 \ll |c_4|^{5/2+5/2\varepsilon} N^{1+\varepsilon}$.

Thus $\qquad |c_4|^{1/2-5/2\varepsilon} \ll N^{1+\varepsilon}$,

or simply $\qquad |c_4|^{1/2} \ll N^{1+\varepsilon}$,

so that $\qquad |c_4|^{1/4} \ll N^{1/2+\varepsilon}$.

The other case, $|c_4|^3 \le |c_6|^2$, similarly gives

$$|c_6|^{1/6} \ll N^{1/2+\varepsilon}.$$

Combining the cases gives

$$\text{SUP}\left(|c_4|^{1/4}, |c_6|^{1/6}\right) \ll N^{1/2+\varepsilon}.$$

PROGRESS REPORT

Rearranging B&SD gives:

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(rank fixed)

Thus, to bound $|\text{Ш}|$, we seek bounds for:

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- $\dfrac{1}{\Omega} \ll N^{1/2+\varepsilon}$, DONE

- $\dfrac{1}{R} \ll C_r$ (constant depending on r),
  TO BE SHOWN

- $\dfrac{1}{\prod_{p|N} c_p} \leq 1$, since each $c_p = \left|\dfrac{E(\mathbf{Q}_p)}{E_0(\mathbf{Q}_p)}\right| \geq 1$

Establishing $\frac{1}{R} \ll C_r$

Recall that the regulator is

$$R = \mathsf{DET}\left[\left\langle P_i, P_j \right\rangle\right],$$

where $\langle \cdot, \cdot \rangle$ is the canonical height pairing, and $\{P_i\}$ is a basis for $E(\mathbf{Q})/E(\mathbf{Q})_{\mathsf{TOR}}$.

Viewing the free $\mathbf{Z}$-module $E(\mathbf{Q})/E(\mathbf{Q})_{\mathsf{TOR}}$ as a lattice (of rank r) inside the $\mathbf{R}$-vectorspace

$$\frac{E(\mathbf{Q})}{E(\mathbf{Q})_{\mathsf{TOR}}} \bigotimes_{\mathbf{Z}} \mathbf{R} \qquad \text{(of dimension } r\text{)},$$

the regulator is simply

$$R = \mathsf{COVOL}(\mathsf{lattice})^2,$$

where volume is measured with respect to the length

$$\ell(P) = \sqrt{\langle P, P \rangle}.$$

Establishing $\frac{1}{R} \ll C_r$

Among all nonzero lattice points, let h_0 be the minimal canonical height. Three results will help us bound $1/R$:

- Minkowski's Theorem from the geometry of numbers will give us:
$$\frac{1}{R} \ll c_r \left(\frac{1}{h_0}\right)^r.$$

- Hindry–Silverman will give us:
$$\frac{1}{h_0} \ll \text{something involving } \mathcal{S}.$$

- Szpiro's conjecture will give us:
$$\mathcal{S} \le 6 + \varepsilon + c_\varepsilon.$$

Establishing $\frac{1}{R} \ll c_r \left(\frac{1}{h_0}\right)^r$

Theorem (Minkowski)
Let B_n denote the ball of radius n. If

$$\mathrm{VOL}(B_n) \geq 2^r\,\mathrm{COVOL}(\text{lattice}),$$

then B_n contains a nonzero lattice point. In particular, the minimal nonzero length ℓ_0 satisfies $\ell_0 \leq n$.

Corollary
Since $\mathrm{VOL}(B_n) = n^r\,\mathrm{VOL}(B_1)$, the theorem requires

$$n^r \geq 2^r\,\mathrm{COVOL}(\text{lattice})/\,\mathrm{VOL}(B_1),$$

and gives ${\ell_0}^r \leq n^r$. In particular, the possibility

$$n^r = 2^r\,\mathrm{COVOL}(\text{lattice})/\,\mathrm{VOL}(B_1)$$

gives us the *unconditional conclusion*

$$\ell_0 \leq 2^r\,\mathrm{COVOL}(\text{lattice})/\,\mathrm{VOL}(B_1).$$

Establishing $\frac{1}{R} \ll c_r \left(\frac{1}{h_0}\right)^r$

The corollary gave,

$$\ell_0 \leq 2^r \,\mathsf{COVOL(lattice)} / \mathsf{VOL}(B_1),$$

which for us becomes

$$\ell_0{}^r \leq 2^r \sqrt{R} / \mathsf{VOL}(B_1).$$

Length and canonical height satisfy the identity $\ell(P)^2 = 2\hat{h}(P)$; in particular, $\ell_0{}^2 = 2h_0$. Thus we have

$$(2h_0)^r \leq 2^{2r} R / \mathsf{VOL}(B_1)^2,$$

so that

$$\frac{1}{R} \leq \frac{2^r / \mathsf{VOL}(B_1)^2}{h_0{}^r}.$$

In short, we have established

$$\frac{1}{R} \ll c_r \left(\frac{1}{h_0}\right)^r.$$

Establishing $\frac{1}{R} \ll C_r$

Theorem (Hindry–Silverman)
Letting $\mathcal{S} = \mathsf{LOG}\,|\mathcal{D}|\,/\,\mathsf{LOG}\,N$ denote the "Szpiro number", we have

$$\frac{1}{h_0} \leq \frac{(20\mathcal{S})^8 10^{1.1+4\mathcal{S}}}{\mathsf{LOG}\,|\mathcal{D}|}.$$

Conjecture (Szpiro)
We have $|\mathcal{D}| \ll N^{6+\varepsilon}$. Hence we have

$$\mathsf{LOG}\,|\mathcal{D}| \leq c_\varepsilon + (6+\varepsilon)\,\mathsf{LOG}\,N,$$

or

$$\mathcal{S} = \frac{\mathsf{LOG}\,|\mathcal{D}|}{\mathsf{LOG}\,N} \leq 6 + \varepsilon + \frac{c_\varepsilon}{\mathsf{LOG}\,N} \leq 6 + \varepsilon + c_\varepsilon.$$

Establishing $\frac{1}{R} \ll C_r$

Now combine the pieces:

$$
\begin{aligned}
\frac{1}{R} &\ll c_r \frac{1}{{h_0}^r} \\
&\leq c_r \frac{(20\mathcal{S})^{8r} 10^{1.1r+4r\mathcal{S}}}{(\mathsf{LOG}\,|\mathcal{D}|)^r} \\
&\leq c_r \frac{(20(6+\varepsilon+c_\varepsilon))^{8r} 10^{1.1r+4r(6+\varepsilon+c_\varepsilon)}}{(\mathsf{LOG}\,|\mathcal{D}|)^r}
\end{aligned}
$$

Since $|\mathcal{D}|$ is always at least 3, we know

$$\frac{1}{\mathsf{LOG}\,|\mathcal{D}|} < 1,$$

and thus we may conclude

$$\frac{1}{R} \ll C_r.$$

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These slides were last modified in March 1999. The most recent version is available for anonymous retrieval from the website:

`www.math.arizona.edu/~aprl`

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