

## SUBGROUPS OF THE SYMMETRIC GROUP

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We started our research with the intent on answering the following question: can we find a way to calculate all the subgroups of the symmetric group. This is easier said than done, as the number of subgroups for a symmetric group grows quickly with each successive symmetric group. This problem can actually be simplified to finding the subgroup conjugacy classes.

So now we have the slightly different question of how to find all the conjugacy classes of a subgroup. One tool that we used to help answer this question is GAP. GAP, Groups, Algorithms, and Programming, is a programming system for computation discrete algebra. One of its commands is “ConjugacyClassesSubgroups” which, when given a symmetric group, calculates its subgroup conjugacy classes. However, if this were all we needed to calculate our conjugacy classes, our paper would be rather short. This command is designed to work for all groups not just the conjugacy groups, and runs rather slowly. In addition, after a certain point, GAP runs out of memory. So we needed to design another program to write into GAP in order to calculate the conjugacy classes of symmetric subgroups.

Before we started our programming, the conjugacy classes of symmetric groups had been found up through  $S_{12}$ . The number of these conjugacy classes were listed online on a very use website entitled “The On-Line Encyclopedia of Integer Sequences.” Through this, we were able to check our results up to  $S_{12}$ . We were also able to discover the conjugacy classes up to  $S_{15}$ , including confirming someone else’s results on the  $S_{14}$ . Given more time and newer computers, we would be to find subgroup conjugacy classes for even larger symmetric groups.

In order to calculate these conjugacy classes, we wrote the program which I will call “ConjClasses.” We will break our process into pieces. First, we will show that if we are given some subgroup of a symmetric group  $S_n$  that it will be conjugate to one of the groups in our list. Secondly, we will show that our list is duplicate-free; that is that there are no conjugate subgroups in our list.

Say we are given some subgroup of the symmetric group  $S_n$ . We first find the length of its smallest orbit, and we call that length  $l$ . We conjugate such that this smallest orbit is  $n$ ’s right-most points. This does not affect our results, as we will compute our subgroups up to conjugation. We then break our subgroup into two subgroups, one acting on  $k$  points and one acting on  $l$  points, where  $k = n - l$ . Let us call a subgroup of  $S_k$   $G$  and the subgroup of  $S_l$  which acts on our smallest orbit  $H$ . We know that  $H$  is a transitive subgroup, as it only acts on one orbit, while  $G$  may be transitive or intransitive. It simplifies matters for us think of our symmetric group  $S_n$  in terms of transitive groups, as GAP has a library listing all the transitive groups up to  $S_{30}$ . We knew that it was unlikely for us to figure out the subgroups for symmetric groups on more than 30 points, so this library was more than enough for our purposes. We can also think of intransitive groups as

direct products of transitive groups, so have broken down our symmetric group in terms of transitive groups.

Our program “ConjClasses” builds on itself. That is, once we have discovered the subgroup conjugacy classes of symmetric group  $S_i$  we store them, and then use these results in our calculation of  $S_{i+1}$ . So we have decided to break  $S_n$  into terms of  $S_k$  and  $S_l$ , both of have smaller degree than  $S_n$ . Thus, for we already have each possible  $G$  and  $H$ . We then calculated using SubDirProd, another program we wrote, all their subdirect products up to conjugation in  $S_k \times S_l$  (more on SubDirProd later). We will thus create a group which already exists in our list of subgroups.

We must now verify that our list of subgroups contains no conjugates. We already know that they are unique up to conjugation by  $S_k \times S_l$ . However, we have not checked if they are conjugate in all of  $S_n$ . We could very well have an element in  $S_n$  which switches an orbit of  $S_k$  with the orbit of  $S_l$ . However, if this happens, we may wind up with a group conjugate to a group already in our list. Thus, we must account for these cases.

In order to simplify this process, we will label the orbits in  $S_k$  and the orbit in  $S_l$  with a key which contains two pieces of information: its orbit length and its TransitiveIdentification number. A TransitiveIdentification number is merely a unique number which GAP has assigned a transitive group. If two groups are conjugate, its orbits must have identical keys. Thus now we only need to check the groups which have identical orbit keys for conjugation. Checking for conjugation is a tedious process, so this labeling saves us time. If we find two groups are conjugate, we keep only one in our list. We can now be sure that our list has unique representatives of all the conjugacy classes of the subgroups of  $S_n$ .

This outlines our process on calculating this conjugacy classes. We will now describe our process of calculating these subdirect products. However, before we do so, it is necessary for us to explore the properties of subdirect products.

By definition,  $P$  is a subdirect product of  $A$  and  $B$  if it is a subgroup of  $A \times B$  where  $P^\alpha = A$  and  $P^\beta = B$  for the projections  $\alpha : A \times B \mapsto A$  and  $\beta : A \times B \mapsto B$ . From this definition, we can derive much about the structure of subdirect products. We will prove this structure, and for clarity please refer to the attached diagram.

$P$  contains the normal subgroups  $\langle \ker\alpha \rangle$  and  $\langle \ker\beta \rangle$ . We can see this as the kernel of a homomorphism is normal in its domain. We can also prove that  $\langle \ker\alpha, \ker\beta \rangle$  is a normal subgroup  $P$ .

Since  $\langle \ker\alpha \rangle \triangleleft P$  and  $\langle \ker\beta \rangle \triangleleft P$ , we know that  $mpm^{-1} \in \ker\alpha$  for all  $p$  in a given subdirect product  $P$  and  $m \in \ker\alpha$ , and that  $vpv^{-1} \in \ker\beta$  for all  $p$  in a given subdirect product  $P$  and  $v \in \ker\beta$ . We know that an element in  $\langle \ker\alpha, \ker\beta \rangle^\alpha$  is of the form  $m_1v_1m_2v_2\dots m_nv_n$  for  $m_i \in \ker\alpha, v_i \in \ker\beta$ . If we conjugate such an element by some  $p \in P$ , we get

$$pm_1v_1m_2v_2\dots m_nv_np^{-1}.$$

If we insert in between each  $m$  and  $n$  element a  $p^{-1}p$ , we get

$$pm_1p^{-1}pv_1p^{-1}pm_2p^{-1}pv_2\dots m_np^{-1}pv_np^{-1}.$$

Thus, we can separate this into

$$(1) \quad (pm_1p^{-1})(pv_1p^{-1})(pm_2p^{-1})(pv_2p^{-1})\dots(pm_np^{-1})(pv_np^{-1}).$$

But since  $\langle \ker\alpha \rangle \triangleleft P$  and  $\langle \ker\beta \rangle \triangleleft P$ , we know that  $pm_i p^{-1} \in \ker\alpha$  and  $pv_i p^{-1} \in \ker\beta$ , so  $1 \in \langle \ker\alpha, \ker\beta \rangle^\alpha$ . Therefore,  $\langle \ker\alpha, \ker\beta \rangle^\alpha \triangleleft P$ . This can be expanded to say that if we have two normal subgroups  $X$  and  $Y$  of a group  $G$ , that  $\langle X, Y \rangle \triangleleft G$ .

Let us now consider the subgroup created by  $\langle \ker\alpha, \ker\beta \rangle^\alpha$ , which we will call  $D$ . This subgroup is also a normal subgroup of  $A$ . Similarly,  $E = \langle \ker\alpha, \ker\beta \rangle^\beta$  is a normal subgroup of  $B$ .

Finally, we can also prove that  $A/D \cong B/E$ . We know if we have a group  $G$  which has a normal subgroup  $H$ , that

$$G/H \cong \gamma G / \gamma H$$

for some homomorphism  $\gamma$ . From this, we can see that

$$P / \langle \ker\alpha, \ker\beta \rangle^\alpha \cong A/D$$

and similarly, that

$$P / \langle \ker\alpha, \ker\beta \rangle^\alpha \cong B/E.$$

By the transitive law, we have

$$A/D \cong B/E$$

So once we have normal subgroups  $D, E$  we can show that the factor groups they produce are isomorphic.

Note: We know that there will  $A$  and  $B$  will always have normal subgroups, even if they are the trivial group.

Thus, we can see that we can derive a lot about the structure of a subdirect product from its seemingly simple definition. With this information, we can see that a subdirect product  $P$  is of the form

$$P = \{(a, b) | (Da)^{r\zeta} = Eb\}, r \in \text{Aut}(A/D).$$

Our process in SubDirProd is based on the following theorem.

**Theorem 1.** *Let  $A \leq S_k$  and  $B \leq S_l$  be permutation groups, so  $A \times B \leq S_k \times S_l$ . Let  $\mathcal{D}$  be a set of representatives of all the normal subgroups of  $A$  up to conjugation by  $N_{S_k}(A)$ , the normalizer of  $A$  in  $S_k$ , and similarly let  $\mathcal{E}$  be a set of representatives of all the normal subgroups of  $B$  up to conjugation by  $N_{S_l}(B)$ .*

*Consider  $g \in N_{N_{S_k}(A)}(D)$  which are the  $\{g \in G | g^{-1}Ag = A \text{ and } g^{-1}Dg = D\}$ . Let  $S \leq \text{Aut}(A/D)$  be the subgroup induced by  $N_{N_{S_k}(A)}(D)$  in the following way: for  $a \in A$  and  $g \in N_{N_{S_k}(A)}(D)$ , we get the automorphism  $Da \mapsto Dg^{-1}ag$ . Let  $\tilde{T}$  be the corresponding subgroup in  $\text{Aut}(B/E)$  induced by  $N_{N_{S_l}(B)}(E)$ .*

*For every pair where  $D \in \mathcal{D}$  and  $E \in \mathcal{E}$  where  $A/D \cong B/E$ , let  $\zeta : A/D \mapsto B/E$  be a fixed isomorphism. Now define  $T = \{\zeta\tau\zeta^{-1} | \tau \in \tilde{T}\}$ , which hence is a subgroup of  $\text{Aut}(A/D)$ . Finally, let  $R$  be the set of representatives for the double cosets  $S \backslash \text{Aut}(A/D) / T$ .*

*Then the set  $\mathcal{P} = \{P_{D,E,r} | D \in \mathcal{D}, E \in \mathcal{E}, r \in R\}$  is the set of all subdirect products  $A$  and  $B$  up to conjugacy in  $S_k \times S_l$ .*

*Proof.* First, we must show that if  $P := P_{D,E,r}$  and  $P' := P'_{D',E',r'}$  are two conjugate subdirect products in  $\mathcal{P}$ , then in fact  $P = P'$ . Secondly, we must show that all subdirect products of  $A \times B$  have a representative in  $\mathcal{P}$  up to conjugation.

Let us assume that  $P, P'$  are conjugate, that is, there exists some  $q \in S_k \times S_l$  such that

$$q^{-1}Pq = P'.$$

Let  $\alpha : A \times B \mapsto A$  and  $\beta : A \times B \mapsto B$ . We need to show that  $D \sim D'$ ,  $E \sim E'$ , and  $SrT = Sr'T$ . We will begin by showing that  $D \sim D'$ .

By definition, we know that  $D = \langle \ker_P \alpha, \ker_P \beta \rangle^\alpha$ . Similarly,  $D' = \langle \ker_{P'} \alpha, \ker_{P'} \beta \rangle^\alpha$ . Let us consider  $k \in \langle \ker_P \alpha, \ker_P \beta \rangle$  where

$$k = a_1 b_1 a_2 b_2 \dots a_n b_n.$$

for  $a_i \in \ker_P \alpha$  and  $b_i \in \ker_P \beta$ . Let us conjugate  $k$  by  $q$  to get

$$q^{-1} k q = q^{-1} (a_1 b_1 a_2 b_2 \dots a_n b_n) q$$

If we place  $q q^{-1}$  in between each  $a_i$  and  $b_i$  of we get

$$q^{-1} k q = q^{-1} a_1 q q^{-1} b_1 q q^{-1} a_2 q q^{-1} b_2 q q^{-1} \dots q q^{-1} a_n q q^{-1} b_n q$$

which is the same as

$$q^{-1} k q = (q^{-1} a_1 q) (q^{-1} b_1 q) (q^{-1} a_2 q) (q^{-1} b_2 q) q^{-1} \dots q (q^{-1} a_n q) (q^{-1} b_n q).$$

However, each  $q^{-1} a_i q \in \ker_{P'} \alpha$  since  $q^{-1} a_i q \in P'$  and

$$(q^{-1} a_i q)^\alpha = q^{-1\alpha} a_i^\alpha q^\alpha = q^{-1\alpha} q^\alpha = (q^{-1} q)^\alpha$$

which is the identity of  $P'$ . Similarly, each  $q^{-1} b_i q \in \ker_P \beta$ . Therefore, when we conjugate an element in  $\langle \ker_P \alpha, \ker_P \beta \rangle$  by  $q$  we get an element in  $\langle \ker_{P'} \alpha, \ker_{P'} \beta \rangle$ . Since  $D = \langle \ker_P \alpha, \ker_P \beta \rangle^\alpha$  and  $D' = \langle \ker_{P'} \alpha, \ker_{P'} \beta \rangle^\alpha$  we now have that  $D \sim D'$ . Since we defined  $\mathcal{D}$  to be the set of representatives of all the normal subgroups of  $A$  up to conjugation by  $N_{S_k}(A)$ . This means that we must also show that  $q \in N_{S_k}(A)$ . This turns out to be trivial, as we assumed that  $P, P'$  are both subdirect products of  $A \times B$ . Therefore, when we conjugate by  $q$ , we are taking elements in  $A$  to other elements in  $A$ . Hence,  $q : D \mapsto D'$ , where  $q \in N_{S_k}(A)$ . Since we had defined  $\mathcal{D}$  to be the set of *representatives* of all the normal subgroups of  $A$  up to conjugation by  $N_{S_k}(A)$ , we in fact have  $D = D'$ .

By a similar process, we can show that  $E = E'$ , as  $q^\beta$  conjugates  $E$  to  $E'$ .

It remains to be shown that  $SrT = Sr'T$ . We define  $T = \{\zeta \tau \zeta^{-1} \mid \tau \in \tilde{T}\}$ , so this is the same as showing  $Sr\zeta \tilde{T} \zeta^{-1} = Sr'\zeta \tilde{T} \zeta^{-1}$ , or  $Sr\zeta \tilde{T} = Sr'\zeta \tilde{T}$ . As  $\zeta$  was a fixed isomorphism which maps  $A/D \mapsto B/E$ , let us choose  $\zeta$  to be the isomorphism such that  $r\zeta = \zeta$ . Now all we need to show is that  $S\zeta \tilde{T} = Sr'\zeta \tilde{T}$ .

Let us remind ourselves what the elements in our subdirect products look like.

$$\begin{aligned} P_{D,E,r} &= \{(a, b) : (Da)^{r\zeta} = Eb\} \\ &= \{(a, b) : (Da)^\zeta = Eb\} \end{aligned}$$

since  $r\zeta = \zeta$ , and

$$P'_{D',E',r'} = \{(a, b) : (Da)^{r'\zeta} = Eb\}$$

Since  $q^{-1} P q = P'$ , we know  $(q^{-1\alpha} a_o q^\alpha, q^{-1\beta} b_o q^\beta) \in P'$  for some  $(a_o, b_o) \in P$ . Therefore, we also have

$$(Dq^{-1\alpha} a_o q^\alpha)^{r'\zeta} = Eq^{-1\beta} b_o q^\beta$$

However,  $S \leq \text{Aut}(A/D)$  is the subgroup induced by  $N_{N_{S_k}(A)}(D)$  where for  $g \in N_{N_{S_k}(A)}(D)$ , we have  $Da \mapsto Dg^{-1}ag$ . We have already shown that  $q^\alpha \in N_{N_{S_k}(A)}(D)$ , so therefore there exists a  $\phi_\alpha \in S$  where  $\phi_\alpha$  is defined by conjugation by  $q^\alpha$ . There exists a similar  $\phi_\beta \in \tilde{T}$  defined by conjugation by  $q^\beta$ . Thus, we can now rearrange our equation

$$(Dq^{-1\alpha} a_1 q^\alpha)^{r'\zeta} = Eq^{-1\beta} b_1 q^\beta$$

to be

$$Da_o^{\phi_\alpha \zeta} = Eb_o^{\phi_\beta}$$

We can bring over the  $\phi_\beta$  to get

$$Da_o^{\phi_\alpha r' \zeta \phi_\beta^{-1}} = Eb_o$$

Since  $(a_o, b_o) \in P$ , we also know that

$$(Da_o)^\zeta = Eb_o$$

We can now see that  $\zeta$  and  $\phi_\alpha r' \zeta \phi_\beta^{-1}$  act identically on  $Da_i$ . Therefore,

$$\zeta = \phi_\alpha r' \zeta \phi_\beta^{-1}.$$

We can now multiply both sides by  $S$  on the left and  $\tilde{T}$  on the right to get

$$S \backslash \zeta / \tilde{T} = S \backslash \phi_\alpha r' \zeta \phi_\beta^{-1} / \tilde{T}.$$

But  $\phi_\alpha \in S$  and  $\phi_\beta \in \tilde{T}$ , so we have

$$S \backslash \zeta / \tilde{T} = S \backslash r' \zeta / \tilde{T},$$

which is what we wanted to show. As above, since  $R$  is the set of representatives for the double cosets  $S \backslash \text{Aut}(A/D) / T$ , we have that  $r = r'$ . Thus, we have shown that  $D = D'$ ,  $E = E'$ , and  $r = r'$ .

Now we must show that all subdirect products of  $A \times B$  have a representative up to conjugation in  $P$ .

Let us consider some subdirect product  $\tilde{P}_{\tilde{D}, \tilde{E}, \tilde{r}}$ , where  $\tilde{D} \in \mathcal{D}$ ,  $\tilde{E} \in \mathcal{E}$ ,  $\tilde{r} \in \mathcal{R}$ . There must be some  $D \in \mathcal{D}$  such that  $D \sim \tilde{D}$ , since by construction  $\mathcal{D}$  contains a representative for all conjugacy classes of normal subgroups of  $A$ . A similar  $E \in \mathcal{E}$  exists for  $\tilde{E}$  by the same argument. Also, there must be some  $r \in R$  for which  $r$  and  $\tilde{r}$  lie in the same double coset  $S \backslash \text{Aut}(A/D) / T$ , since the cosets partition the group  $\text{Aut}(A/D)$ . Hence, since each subdirect product  $P$  is determined by its  $D, E$ , and  $r$ , then  $\tilde{P}_{\tilde{D}, \tilde{E}, \tilde{r}} \sim P_{D, E, r}$  for some other subdirect product  $P_{D, E, r} \in \mathcal{P}$ . Thus, we have shown all we need to show for our proof.  $\square$

With this theorem supporting us, we can now review our process in SubDirProd.

SubDirProd is a program that calculates all possible subdirect products of the groups  $A$  and  $B$ . We approach this in two parts; we will first set limitations to our subgroups to ensure that we are in fact dealing within the constraints of a subdirect product. Secondly, we will go through the process of actually generating a subdirect product.

We input groups  $A \in S_k$  and  $B \in S_l$ . We have the projection map  $\alpha$  which maps  $A \times B \mapsto A$ , and the projection map  $\beta$  which maps  $A \times B \mapsto B$ . We then consider the normal subgroups  $D \leq A$  up to conjugation by  $N_{S_k}(A)$  and the normal subgroups  $E \leq B$  up to conjugation by  $N_{S_l}(B)$  where  $A/D \cong B/E$ . We now wish to consider all possible isomorphisms which will not result in conjugate subgroups. Thus, we take a fixed isomorphism  $\zeta$  and apply to it all the automorphisms of  $A/D$  which are the representatives of the double cosets  $S \backslash \text{Aut}(A/D) / T$ . We now have all the tools and restrictions necessary to generate our subdirect product.

It is important to note that GAP stores all its groups by their generating elements, so we will therefore define our subdirect product by its generating elements. To determine these, we will need to generate two groups:  $\ker \alpha$  and  $A$ .

We know that  $\ker\alpha = \{(a, b) \mid (a, b)^\alpha = I_A\}$ . It is clear that our first coordinate  $a$  must itself be  $I_A$ . However, we cannot choose any arbitrary element in  $B$  for our  $b$ . This is because of the restrictions that we previously named. Say we map an element which is in the  $\ker\alpha$  to  $A$ . We can see that we end up with the  $I_A$ . Take the natural homomorphism to create  $D_{I_A}$ , which is merely  $D$ . We now apply an isomorphism to get an element in  $B/E$ . Isomorphisms preserve identities as well, so when we apply our isomorphism to  $D$  we will get  $E$ . However, according to our basic properties of subdirect products, we also have the equation:

$$P = \{(a, b) \mid (Da)^{r^\zeta} = Eb\}$$

where  $r$  is one of our specified automorphisms. Thus, we have taken an ordered pair  $(a_1, b_1)$  where  $(Da_1)^{r^\zeta} = E$ . However, from we also know that

$$(Da_1)^{r^\zeta} = Eb_1.$$

Therefore,

$$E = Eb_1$$

which implies that

$$b_1 \in E.$$

But  $b_1$  was merely our second tuple in our element in  $\ker\alpha$ . Thus, all elements in the  $\ker\alpha$  must be of the form  $(I_A, e)$  where  $e$  is a generator of  $E$ . We now construct  $A$ . This is rather simple; we must take the elements  $(a, b) \in P$  where  $a$  is in the generators of  $A$  and  $b \in B$  which satisfy  $(Da)^{r^\zeta} = Eb$ .

From these two elements,  $\ker\alpha$  and  $A$ , we can construct our subdirect product. If we refer once again to our diagram, we can see that we generate everything in our subdirect product as parallel lines map isomorphically.

Thus, we have formed our subdirect products for  $A \times B$ . We can consult our theorem to convince ourselves that we truly have all our subdirect products up to conjugation in  $S_k \times S_l$ .

We can now see that with our two programs, `ConjClasses` and `SubDirProd`, we can construct all the subgroup conjugacy classes of the symmetric groups, though not without upper limits (as we mentioned before, we only have information for transitive groups up to  $S_{30}$ ). We would finally like to mention a technique which we used while programming. While our programs greatly simplify the process of calculating conjugacy classes, we are still dealing with very large numbers and many calculations. Therefore, our programs still run for a significant amount of time, and it became useful for us to try to find ways to reduce this time. One such technique we used was parallelization, particularly during the process of calculating subdirect products. Unlike the process of calculating conjugacy classes, this does not require prior knowledge any other subdirect products. Hence, we decided to break up the process of calculating these subdirect products onto multiple machines, with several machines working on harder cases and fewer machines working on simpler cases.

Thus, through a process of breaking down and simplifying into terms we could work with, we were able to calculate the subgroups of the symmetric group  $S_{15}$ . The number of conjugacy classes of  $\{S_1 \dots S_{12}\}$  are as follows:

$$\{1, 2, 4, 11, 19, 56, 6, 296, 554, 1593, 3094, 10723\}$$

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