

# Prescribing Curvature on Triangulated 3-Manifolds

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## Introduction

Let  $\mathcal{T} = (V, E, F, T)$  be a triangulated 3-manifold. Definitions:

- Edge Curvature:  $K_e = (2\pi - \sum_{e < t} \beta_{e,t}) \ell_e$
- Vertex Curvature:  $\frac{1}{2} \sum_{v < e} K_e$

where  $\beta_{e,t}$  is the dihedral angle about edge  $e$  in tetrahedron  $t$ ,  $\ell_e$  is the length of  $e$ , and  $e < t$  denotes iteration over tetrahedra containing  $e$ .

## The Pentachoron

The pentachoron has the combinatorial structure of the boundary of the 4-simplex.

- The 1-skeleton of the pentachoron is the complete graph  $K_5$
- The pentachoron has F vector  $(5, 10, 10, 5)$  (i.e. 5 vertices, 10 edges, 10 faces, 5 tetrahedra).
- As a triangulation, the pentachoron is a compact 3-manifold homeomorphic to  $S^3$

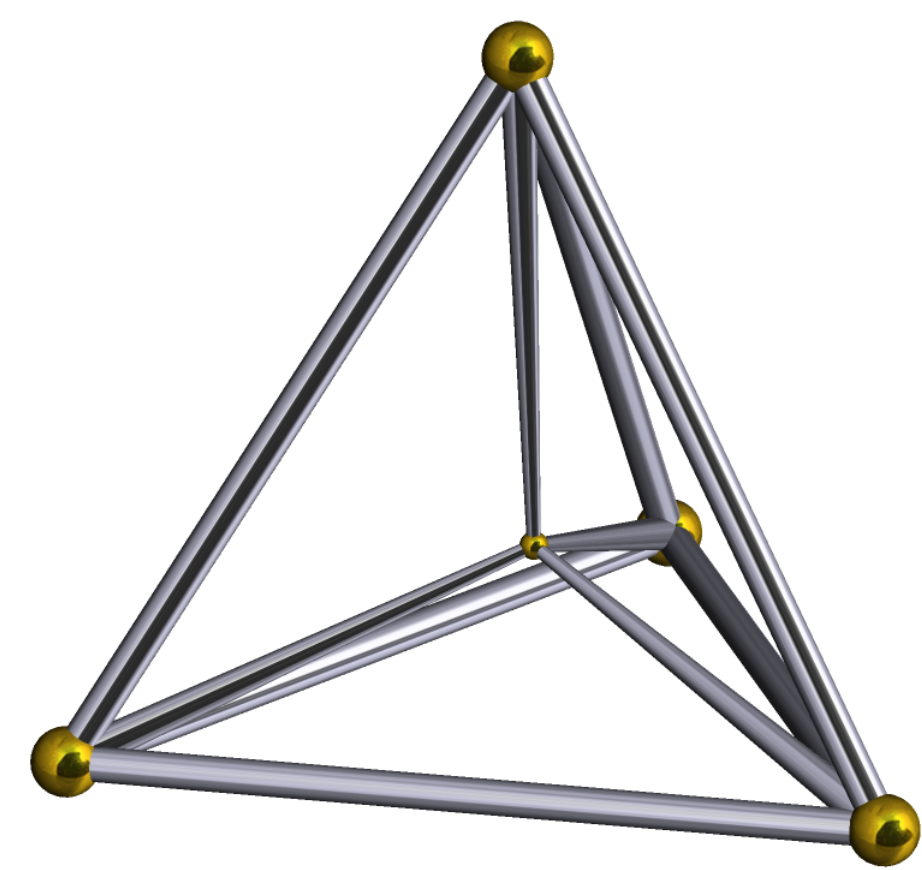


Figure: A projection of the pentachoron

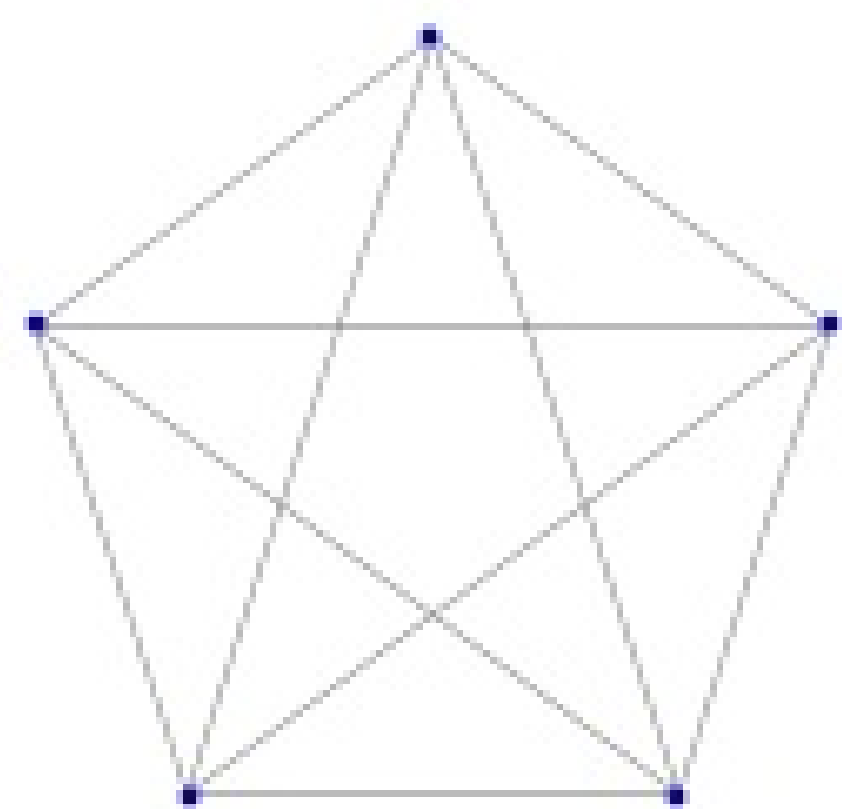


Figure: The pentachoron's 1-Skeleton  $K_5$

## Edge Curvature

**Theorem:** Let  $\mathcal{T}(V, E, F, T)$  be a triangulated 3-manifold with negative edge curvature for all edges. Then

$$|E| > 3|V|.$$

Proof requires the following results:

- Euler-Poincare Theorem:  $|V| - |E| + |F| - |T| = 0$ .
- Gaddum [1]: if  $\beta_1, \dots, \beta_6$  are the dihedral angles of a tetrahedron, then  $2\pi \leq \sum_{i=1}^6 \beta_i \leq 3\pi$ .

**Corollary:** The pentachoron does not admit a metric with negative edge curvature at each edge.

**Proof:** The pentachoron has  $|V| = 5$  and  $|E| = 10$ .  $\square$

## Vertex Curvature

**Conjecture:** The pentachoron does not admit a metric with negative vertex curvature for each vertex.

- This question is more complicated than the corresponding result for edge curvature because some edges may have negative curvature.
- On the pentachoron, after normalizing by setting  $\sum_i \ell_i = 1$ ,  $\sum_v K_v$  is a nine variable function.

## References

- [1] J.W. Gaddum, *The Sums of the Dihedral and Trihedral Angles in a Tetrahedron*, Amer. Math. Month. (1952), **59**, no. 6, 70–371.
- [2] D. W. Walkup, *The Lower Bound Conjecture for 3- and 4-manifolds*, Act Math. **125** (1970), no. 1, 75–107.
- [3] Marsaglia, G, *Choosing a Point from the Surface of a Sphere*, Ann. Math. Stat. (1972) 43: 645D646
- [4] Glickenstein, D. *Discrete conformal variations and scalar curvature on three dimensional manifolds* arXiv:0906.1560v1 [math.DG]

## A Cyclic Length Metric

**Definition:** A cyclic length metric on the pentachoron is one where each vertex is incident to edges with the same set of lengths.

- A cyclic length metric imposes more symmetry, specifically:
  - Only two distinct lengths, with two edges of each length incident to each vertex
  - One type of tetrahedron up to isometry

**Theorem:** A cyclic length metric on the pentachoron has positive vertex curvature for each vertex.

- Simplifies  $\sum_v K_v$  into a one variable problem after normalization.
- Minimum of  $\sum_v K_v$  is well above 0.
- Minimum occurs at equal length metric

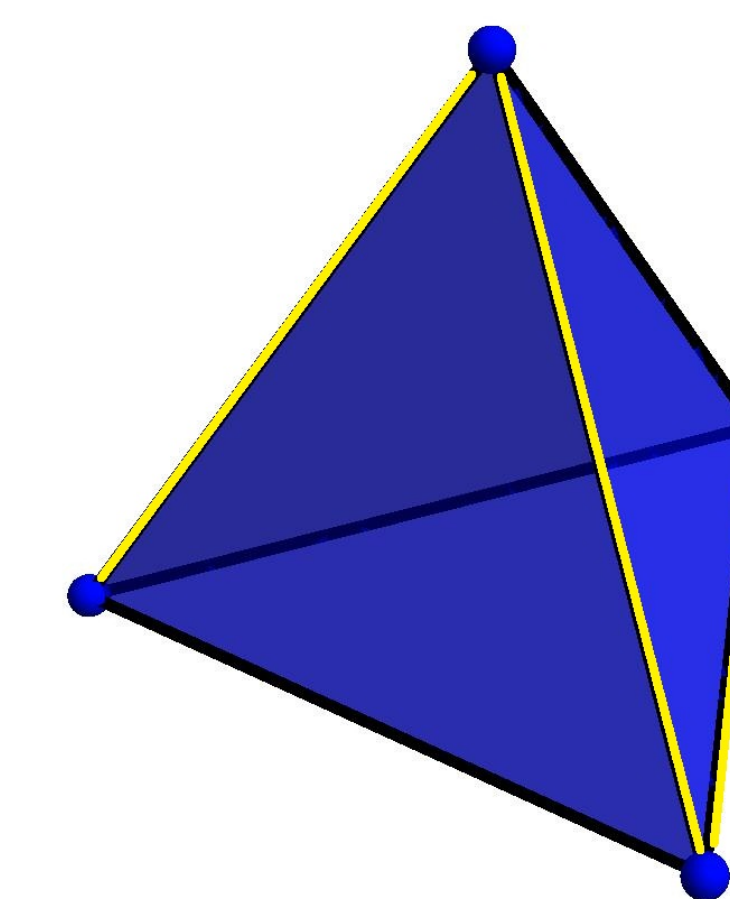
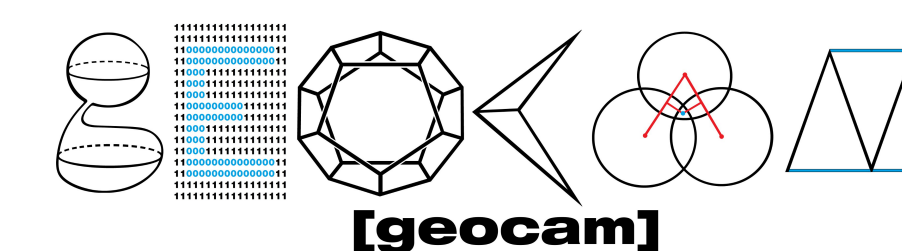


Figure: A Tetrahedron in the cyclic length metric pentachoron

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## Simulations

We randomly sampled pentachora to find empirical bounds for edge and vertex curvatures. As the boundary of a 4-simplex, any pentachoron can be inscribed in a 3-sphere. Hence we took the 3-sphere as our sample space, picking points uniformly by using the following algorithm of Marsaglia [3]:

- Pick  $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$  with each coordinate normally distributed.
- Normalize the vector  $\mathbf{x}$  by the transformation  $\mathbf{x} \mapsto \frac{\mathbf{x}_i}{\|\mathbf{x}\|}$ .
- Pick five points according to this method to create a pentachoron.

We sampled 10,000 pentachora using the above method. Below are frequency plots of all the associated edge and vertex curvatures.

