Mathematics of Source Unfolding
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Arizona Summer Program 2010

Introduction
In studying manifolds, it is desirable to be able to “step inside” of them and view them as a "native creature" would. By approximating smooth manifolds by piecewise flat manifolds that preserve topological information, we are able to use a source unfolding algorithm to create a visualization that reflects topological properties of the original manifold.

Piecewise Flat Manifolds
• An \( n \)-dimensional manifold is a space that is locally homeomorphic to \( \mathbb{R}^n \).
• A piecewise flat manifold consists of some number of \( n \)-dimensional facets connected along their ridges.
  • Exactly two facets are connected at each ridge.
  • At lower dimensional boundaries facets must make a sphere.
• Each facet can be embedded in \( \mathbb{R}^n \).
• Transition maps connecting facets along ridges are encoded using affine transformations.

Affine Transformations
An affine transformation is a linear transformation followed by a translation, i.e., we can represent an affine transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) by the expression \( T x = Ax + b \), where \( A \) is a linear transformation and \( b \) is a vector.

Definitions
- **Facet**: An \( n \)-facet is a finite intersection of closed half spaces in \( \mathbb{R}^n \) with non-zero volume.
- **Ridge**: A codimension 1 facet.
- **Geodesic**: A geodesic is a locally shortest path between two points.
- **Tangent Space of a Point**: An \( n \)-dimensional plane that contains the facet upon which a point from the manifold lies.
- **Visible**: A point defined by a vector in the tangent space is visible if the image of the vector under the exponential map contains no points on facets of codimension two or larger.
- **Blocked Line of Sight**: A line of sight that passes through non-visible points.

Exponential Map and Development
Given a point \( p \) on a manifold \( M \) and a vector \( v \) in its tangent space \( T \), the exponential map returns the point on \( M \) determined by moving a distance \( \| v \| \) along the geodesic through \( p \) in the initial direction of \( v \) for the distance equal to the magnitude of \( v \). The point \( p \) is called a **source point**.

The inverse relation of this map, the development, carries points on \( M \) to (multiple) points in \( T \).

We have constructed a source unfolding map that carries facets and parts of facets from \( M \) to \( T \). The source unfolding is a model of the development relation.

Frustums
A frustum determines part of the visual field in \( T \). It can be defined either as a set of vectors in \( T \) that cross the same list of edges when scaled by a parameter \( c > 0 \), or as the intersection of half-spaces in \( T \) which intersect at the source point.

Results
• By implementing the source unfolding algorithm in JAVA, we were able to produce images depicting the visual field of 2-dimensional piecewise flat manifolds.

Source unfolding of a 40-faced pyramid.

• Theorem: Blocked lines of sight are dense in the lines of sight emanating from any source point.

Acknowledgements
This poster was mentored by David Glickenstein and Danny Maienschein, whose help is acknowledged with much appreciation.

This material is based upon work supported by the National Science Foundation under VIGRE Grant No. 0602173.