THE RICCI BRACKET FLOW FOR SOLVABLE LIE GROUPS

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Abstract. The Ricci bracket flow is a geometric evolution on Lie algebras which is related to the Ricci flow on the corresponding Lie group. For nilpotent Lie groups, these two flows are equivalent. In the solvable case, it is not known whether they are equivalent. We examine a family of solvable Lie algebras and identify various elements of that family which are solitons under the Ricci bracket flow. We also examine behavior of several classes of non-soliton brackets under the flow.

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1. Introduction

The bracket flow is a means of evolving the geometry on a Lie group with left-invarient metric by changing the structure of the bracket of the corresponding Lie algebra. It is related to the Ricci flow. In the case of nilpotent Lie groups, the Ricci flow and the bracket flow are equivalent. It is unknown whether this is true for solvable Lie groups. We work with a specific family of solvable Lie groups and examine its evolution under the bracket flow. We identify solitons use computer experiments which suggest which of these solitons are attractors.

The general form of the brackets we examined is

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$[e_4, \cdot] = \begin{pmatrix} a & b & 0 & 0 \\ -b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We found that the form of these brackets is preserved under the bracket flow.
In fact, we found that the more general bracket form:

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$[e_4, \cdot] = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is preserved under the bracket flow.

For brackets of the first form mentioned, we found that the only solitons are brackets of the forms:

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$[e_4, \cdot] = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[e_4, \cdot] = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[e_4, \cdot] = \begin{pmatrix} a & a & 0 & 0 \\ -a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[e_4, \cdot] = \begin{pmatrix} a & -a & 0 & 0 \\ a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Though it is not of the form primarily studied, we also showed that the bracket

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$[e_4, \cdot] = \begin{pmatrix} a & b & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a soliton under the bracket flow.

We performed some computer experiments which suggested to us that the first and second soliton bracket described (of the four above) were attractive fixed points under the normalized bracket flow and that the others were repelling.

2. Background

The objects studied here are solvable Lie groups with left-invariant metrics.

Definition 1. A **Lie group** is a manifold which also has a group structure (i.e. every point on the manifold is an element of a group) such that the operations of group multiplication and inverse are smooth.

Definition 2. A Lie group G is said to have a **left-invariant metric** if

$$\langle v, w \rangle_g = \langle dL_{g^{-1}}(v), dL_{g^{-1}}(w) \rangle_e$$

Instead of studying the traditional Ricci flow on the Riemannian manifold structure of the Lie group, we instead studied a related flow on the Lie Algebra.

Definition 3. A **Lie Algebra** is a vector space V and a map $[\cdot, \cdot]\colon V \times V \to V$ called the **Lie Bracket** satisfying the following:

(1) The Lie bracket is bilinear.
(2) The Lie bracket is anti-symmetric
(3) $\forall \quad X, Y, \text{and} Z$ in the Lie algebra, the Lie bracket satisfies the following identity, called the **Jacobi Identity**
$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

We are concerned in particular with the set of **solvable** Lie groups.

Every Lie group has a corresponding Lie algebra; the vector space component is the tangent space at the identity of the Lie group, and the bracket is derived from the group multiplication. For all n-dimensional Lie groups G, T_eG is isomorphic to $\mathbb{R}^n$, so the geometry of the space is encoded not in the vector space component of the Lie algebra, but rather in the bracket. If we pick the inner product on T_eG to be the standard inner product on $\mathbb{R}^n$, we can establish a left-invariant metric on the entire Lie group. To describe geometric evolution on the Lie group, instead of evolving the inner products on the tangent spaces, we can fix the inner product as the standard inner prodcut and evolve the bracket. To do this, it is useful to encode the bracket as a finite set of real numbers called the **structure constants**.

Definition 4. For an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the Lie algebra, the Lie bracket can be encoded in the following way:
Since the bracket is bilinear, the bracket of two basis elements will be a linear combination of the n basis elements. So we write

$$[e_i, e_j] = \sum_{k=1}^{n} c_{i,j}^k e_k$$

The real numbers $\{c_{ij}^k\}$ are called the **structure constants**.

Note that due to the anti-symmetry of the bracket, we have that $c_{ij}^k = -c_{ji}^k$

We wish to express the geometric evolution of the Lie group in terms of an evolution through the space of Lie brackets. Since the structure constants encode the bracket, we can use a system of differential equations on the structure constants to describe this flow.

The differential equations which describe the flow of the structure constants through the space of Lie brackets are given by the equation

$$(c_{i,j}^k)' = -\sum_{\ell=1}^{n} (c_{i,j}^\ell \operatorname{Ric}_{k,\ell} - c_{\ell,j}^k \operatorname{Ric}_{\ell,i} - c_{i,\ell}^k \operatorname{Ric}_{\ell,j})$$

Where $\operatorname{Ric}_{a,b}$ is given by

$$\begin{aligned}\operatorname{Ric}_{a,b} = &-\frac{1}{2}\sum_{i=1}^{n}\langle [e_i, e_a], [e_i, e_b]\rangle - \frac{1}{2}\sum_{i=1}^{n}\langle [e_a, [e_b, e_i]], e_i\rangle \\ &+\frac{1}{2}\sum_{1\le i<j\le n}\langle [e_i, e_j], e_a\rangle\langle [e_i, e_j], e_b\rangle\end{aligned}$$

Definition 5. Let $[\mathfrak{g}, \mathfrak{g}] = \{[g, h] | g, h \in \mathfrak{g}\}$. Define the following sequence recursively. let $\mathfrak{g}^{(1)} = \mathfrak{g}$ and let $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$. A Lie algebra is called nilpotent if the sequence $(\mathfrak{g}^{(i)})$ terminates (at the identity) after a finite number of steps.

For nilpotent Lie groups, it has been proven that the bracket flow and the Ricci flow are equivalent [Lau10]. The relationship between the bracket flow and the Ricci flow on solvable Lie algebras is not well-understood. Our goal is to examine some examples of solvable Lie groups.

Definition 6. Given a Lie algebra $\mathfrak{g}$, let $[\mathfrak{g}, \mathfrak{g}] = \{[g, h] | g, h \in \mathfrak{g}\}$. We say a Lie group is solvable if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Our work was primarily focused on looking at a specific family of solvable Lie groups whose form is preserved under the bracket flow and trying to find solitons in this family the bracket flow. A **soliton** here is a point in the space of Lie brackets where the integral curve through that point is a line through the origin.

We will represent integral curves in the space of brackets in the following way:

$$\mu(t) = (c_{1,1}^1(t), c_{2,1}^1(t), \dots, c_{4,4}^4(t))$$

In addition to examining the flow described by the differential equations above, we also want to consider the **normalized flow**, the solutions to which are solutions to the bracket flow projected onto a sphere. Note that since solitions under the unnormalized flow are lines through the origin, under the normalized flow they become fixed points. The normalized bracket flow is given by differentiating the equation

$$\lambda = \frac{\mu}{|\mu|}|\mu(0)|$$

Which gives:

$$\lambda'(0) = |\mu(0)| \left(\frac{\mu'(0)}{|\mu|} + \mu(0) \frac{\mathrm{d}}{\mathrm{d}\,t}\bigg|_{t=0} \frac{1}{|\mu|} \right) =$$

$$\lambda'(0) = |\mu(0)| \left(\frac{\mu'(0)}{|\mu|} + \mu(0) - \mu(0)^{-2} \frac{1}{2\sqrt{\sum_{i,j,k}(c_{ij}^k)^2}} \sum_{i,j,k} 2(c_{ij}^k)(c_{ij}^k)' \right) =$$

$$\lambda'(0) = |\mu(0)| \left(\frac{\mu'(0)}{|\mu|} - \frac{\sum_{i,j,k}(c_{ij}^k)(c_{ij}^k)'}{|\mu(0)|^3} \right) =$$

$$\lambda'(0) = \mu'(0) - \frac{\sum_{i,j,k}(c_{ij}^k)(c_{ij}^k)'}{|\mu(0)|^2}$$

3. Calculating the Bracket flow for Specific Brackets

Hyperbolic space can be described as a Lie group with corresponding Lie algebra characterized by the following bracket structure:

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$[e_4, \cdot] = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for certain real values of a. What this means in terms of the structure constants is that $c_{4,1}^1 = c_{4,2}^2 = c_{4,3}^3 = a$ and $c_{ij}^k = 0$ otherwise.

Motivated by this example, we decided to first study how brackets of the following form behaved under the bracket flow:

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$\begin{pmatrix} c_{4,1}^1 & 0 & 0 & 0 \\ 0 & c_{4,2}^2 & 0 & 0 \\ 0 & 0 & c_{4,3}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system of differential equations for this bracket can be explicitly solved, and solving reveals that this is a soliton under the bracket flow.

After looking at brackets where the only nonzero entries are on the diagonal, we consider brackets of the form

$$[e_i, \cdot] = 0 \text{ for all } i \neq 4$$

$$\begin{pmatrix} c^1_{4,1} & c^1_{4,2} & 0 & 0 \\ c^1_{4,2} & c^2_{4,2} & 0 & 0 \\ 0 & 0 & c^3_{4,3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Results

Proposition 4.1. *Four-dimensional Lie algebras with basis $\{e_1, e_2, e_3, e_4\}$ for which e_1, e_2, and e_3 commute and for which the matrix of the map $[e_4, \cdot]$ is as follows:*

$$\begin{pmatrix} c^1_{41} & c^1_{42} & 0 & 0 \\ c^2_{41} & c^2_{42} & 0 & 0 \\ 0 & 0 & c^3_{43} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

will remain in this form under the bracket flow.

Proof. Using Mathematica and the differential equations stated above for the structure constants under the bracket flow, we find that the derivative is zero for each zero structure constant. Thus we may reduce the system of differential equations for structure constants to the system of five differential equations for c^2_{41}, c^1_{42}, c^2_{41}, c^2_{42}, and c^3_{43}. By existence of solutions to smooth first-order differential equations [Arn92], a solution to this sytem, that is, a solution in which every structure constant besides these five remains zero, must exist through all starting points. However, by uniqueness of solutions [Arn92], no other solution can exist through a given starting point. We have shown that the only solution that exists through each point of this form under the bracket flow preserves this form. □

Proposition 4.2. *Lie algebras for which $[e_4, \cdot]$ is the following map:*

$$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and which are otherwise as described in Proposition 4.1 are solitons under the bracket flow.

Proof. We first calculate that when $c^2_{41} = c^1_{42} = 0$, as they do here, that $(c^2_{41})' = (c^1_{42})' = 0$. Thus, by the same reasoning as in Proposition 4.1, this form is preserved under the bracket flow.

The differential equations arising from the bracket flow are the following.

$$x' = -x(x^2 + y^2 + z^2)$$
$$y' = -y(x^2 + y^2 + z^2)$$
$$z' = -z(x^2 + y^2 + z^2)$$

Setting aside the degenerate case in which $x = y = z = 0$, we have three possibilities: either two of $x(0)$, $y(0)$, and $z(0)$ is zero, or one is, or they are all nonzero. In the first case, assume without loss of generality that $x(0) \neq 0$ and that $y(0) = z(0) = 0$. Then $y'(0) = z'(0) = 0$ and so y and z are always zero. Clearly, then, the solution lies on a line, namely the x-axis.

In the second case, assume without loss of generality that $x(0) \neq 0$, $y(0) \neq 0$, and $z(0) = 0$. Then z is uniquely zero and our system reduces to

$$x' = x(x^2 + y^2)$$
$$y' = y(x^2 + y^2)$$

Note that $(x^2 + y^2) = \frac{x'}{x} = \frac{y'}{y}$. Thus $x'y = y'x$, so $x'y - y'x = 0$. But an application of the product rule shows that the derivative of $\frac{x}{y}$ is given by $\frac{x'y - y'x}{y^2}$. Since $x'y - y'x = 0$, then, the derivative of $\frac{x}{y}$ is 0 and $\frac{x}{y}$ is constant. But saying that the ratio of x to y is constant is equivalent to saying that the solution is a line through the origin, as required.

In the third case, we find by completely analogous reasoning that $\frac{x}{y}$, $\frac{x}{z}$, and $\frac{y}{z}$ are constants. This can again be interpreted geometrically by saying that $(x(t), y(t), z(t))$ follows a line through the origin. Thus in any case, the Lie algebra in question is a soliton. □

Remark 4.3. The preceding proof technique is easily generalized: A Lie algebra is a soliton if $(c_{ij}^k)' c_{lm}^n - c_{ij}^k (c_{lm}^n)' = 0$ for all structure constants c_{ij}^k and c_{lm}^n.

Proposition 4.4. *Members of the following subsets of the general family of brackets described in Proposition 4.1 are solitons under the bracket flow. (We describe the bracket by giving the matrix of* $[e_4, \cdot]$ *in each case.)*

$$\begin{pmatrix} a & b & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & a & 0 & 0 \\ -a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & -a & 0 & 0 \\ a & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proof. The proof is a simple matter of using Mathematica to calculate $(c_{ij}^k)' c_{lm}^n - c_{ij}^k (c_{lm}^n)'$ for all choices of structure constants c_{ij}^k and c_{lm}^n. These values are always zero for all the cases listed above, which proves that Lie algebras with brackets of those forms are solitons under the bracket flow. □

Proposition 4.5. *Lie algebras in which e_1, e_2, e_3 commute and in which $[e_4, \cdot]$ is the following map:*

$$\begin{pmatrix} a & b & 0 & 0 \\ -b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are not *solitons unless they are of one of the forms mentioned in Proposition 4.2 or Proposition4.4.*

Proof. The proof is by contradiction. Assume that a Lie algebra is of the form given above but not of one of the more specific forms discussed in Proposition 4.2 or Proposition 4.4. Saying that the algebra is a soliton is equivalent to saying that it is a fixed point of the normalized bracket flow. We use Mathematica to calculate the differential equations for the normalized bracket flow. They are:

$$a' = \frac{2b^2(-ab^2 - 2a^2d + b^2d + ad^2 + d^3)}{a^2 + b^2 + ad + d^2}$$
$$b' = \frac{-b(a-d)^2(a^2 - b^2 + ad + d^2)}{a^2 + b^2 + ad + d^2}$$
$$d' = \frac{2b^2(a-d)(a^2 + b^2 + 2ad}{a^2 + b^2 + ad + d^2}$$

If we had $b = 0$, the bracket would be of the form discussed in 4.2. If we had $a = d$, then the bracket would be of the second form discussed in 4.4. Thus we may assume that $b \neq 0$ and $a \neq d$. Thus $a' = b' = d' = 0$ is true if the following three equations hold:

$$-ab^2 - 2a^2d + b^2d + ad^2 + d^3 = 0 \tag{1}$$
$$a^2 - b^2 + ad + d^2 = 0 \tag{2}$$
$$a^2 + b^2 + 2ad = 0 \tag{3}$$

By (3), $a^2 = -b^2 - 2ad$. Substituting this expression for a^2 into (2), we find that $-2b^2 - ad + d^2 = 0$, so $d^2 = 2b^2 + ad$. Then (1) yields

$$\begin{aligned} 0 &= -ab^2 - 2a^2d + b^2d + ad^2 + d^3 \\ &= -ab^2 - 2a^2d + b^2d + a(ad + 2b^2) + d(ad + 2b^2) \\ &= -ab^2 - 2a^2d + b^2d + a^2d + 2ab^2 + ad^2 + 2b^2d \\ &= ab^2 - a^2d + 3b^2d + ad^2 \\ &= ab^2 - a^2d + 3b^2d + a(ad + 2b^2) \\ &= ab^2 - a^2d + 3b^2d + a^2d + 2ab^2 \\ &= 3(a+d)b^2 \end{aligned}$$

Since we know that $b^2 \neq 0$, this gives

$$a = -d \tag{4}$$

and (3) then yields $a^2 = b^2$. Either $a = b$, in which case $d = -b$ by (4), or $a = -b$, in which case $d = b$. These two cases are, however, brackets of the forms mentioned in Proposition 4.4, which contradicts our initial assumption. □

5. Computer experimentation

In this section we restrict our discussion to the case in which $[e_4, \cdot]$ is a map of the form

$$\begin{pmatrix} a & b & 0 & 0 \\ -b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because under the normalized bracket flow the norm of the bracket does not change, we fix the norm of the bracket at some arbitrary value in order to reduce the number of variables, enabling us to graph a system of differential equations in two dimensions. For example, assume that the norm of the bracket is equal to 2. Then

$$a^2 + b^2 + (-b)^2 + d^2 + (-a-d)^2 = 2$$

Manipulating this expression, we find that

$$b^2 = 1 - a^2 - d^2 - ad$$

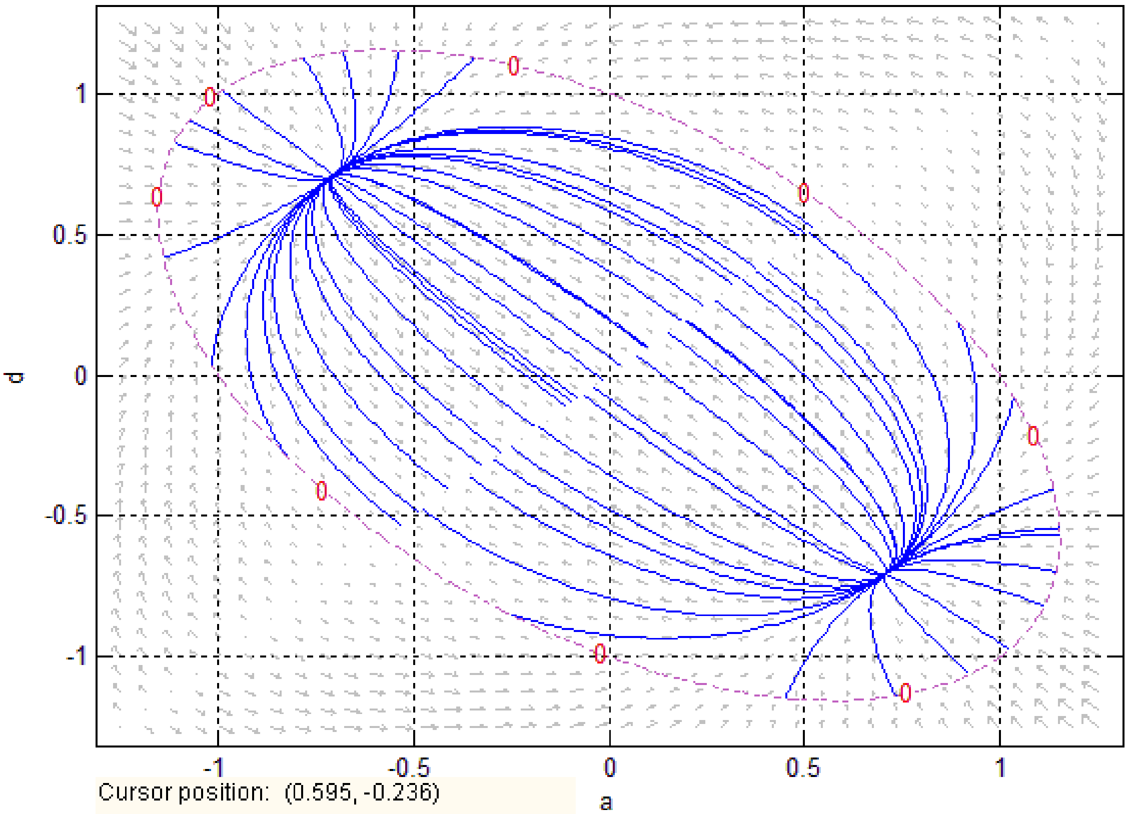
The system of differential equations for the normalized bracket flow is:

$$a' = \frac{2b^2(-ab^2 - 2a^2d + b^2d + ad^2 + d^3)}{a^2 + b^2 + ad + d^2}$$
$$b' = \frac{-b(a-d)^2(a^2 - b^2 + ad + d^2)}{a^2 + b^2 + ad + d^2}$$
$$d' = \frac{2b^2(a-d)(a^2 + b^2 + 2ad)}{a^2 + b^2 + ad + d^2}$$

Substituting our expression for b^2 into the differential equations for a and d and simplifying, we arrive at the following system of two differential equations.

$$a' = -2(-1 + a^2 + ad + d^2)(a^3 + d - 2a^2d + a(-1 + d^2)$$
$$d' = -2(a-d)(1 + ad - d^2)(-1 + a^2 + ad + d^2)$$

We may use a computer to plot the vector field corresponding to this system, and then to numerically generate solution curves and plot them.



We are only interested in the region bounded by the pink ellipse: points outside of this region are extraneous because they correspond to non-real values of b. The pink ellipse is the set of points where $b = 0$. Blue curves are solution curves to the bracket flow. Since, by Proposition 4.2 and 4.4, we have solitons when $a = d$ and when $b = 0$, this picture suggests that non-soliton brackets of the form we are considering always approach a soliton under the bracket flow.

To create this picture, we let the norm of the bracket equal 2. If we set the norm to any positive real number, we achieve scaled versions of the same picture. This leads one to believe that the observed behavior does not depend on the norm of the bracket, but is true in all cases. Thus we formulate the following conjecture.

Conjecture 5.1. *As time approaches infinity, any Lie bracket of the form we are considering with the map* $[e_4, \cdot]$ *represented by the matrix*

$$\begin{pmatrix} a & b & 0 & 0 \\ -b & d & 0 & 0 \\ 0 & 0 & -a-d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

approaches a soliton.

The authors have not successfully proven this conjecture.

Remark 5.2. If it is true, this conjecture implies that all solution curves in our system approach fixed limits as time approaches infinity.

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John Polking, whose free program PPLANE8 we used to create the pictures (one of which is reproduced above) that led us to formulate our main conjecture.

Appendix A. Mathematica code

The following functions were used to calculate Ricci curvature and to generate the differential equations corresponding to the bracket flow on various families of Lie algebras.

```
(* ::Package:: *)

Package["BracketFlow`"]
ric[c_,x_,y_] := Module[{n}, n = Length[x];
-1/2*Sum[x[[l]]*y[[m]]*c[[i]][[l]][[k]]*c[[i]][[m]][[k]],
{i,1,n},{l,1,n},{m,1,n},{k,1,n}]
-1/2 * Sum[x[[l]]*y[[m]]*c[[m]][[i]][[k]]*c[[l]][[k]][[i]],
{i,1,n},{l,1,n},{m,1,n},{k,1,n}]
+1/2*Sum[Sum[(Sum[x[[l]]*c[[i]][[j]][[l]],{l,n}])*
(Sum[y[[l]]*c[[i]][[j]][[l]],{l,n}]),{i,1,j-1}],{j,1,n}]]

ourCase := {{{0,0,0,0},{0,0,0,0},{0,0,0,0},{-a,-c,0,0}},
{{0,0,0,0},{0,0,0,0},{0,0,0,0},{-b,-d,0,0}},
{{0,0,0,0},{0,0,0,0},{0,0,0,0},{0,0,a+d,0}},
{{a,c,0,0},{b,d,0,0},{0,0,-a-d,0},{0,0,0,0}}}

symmetricCase := ourCase/.c->b

generalizedSkewSymmetricCase := ourCase/.c->-b

GetBracketFlowEquation[c_,i_,j_,k_,n_] :=
c[[i]][[j]][[k]]'== Simplify[Sum[
c[[i]][[j]][[l]]*ric[c,UnitVector[n,k],UnitVector[n,l]]
-c[[l]][[j]][[k]]*ric[c,UnitVector[n,l],UnitVector[n,i]]
-c[[i]][[l]][[k]]*ric[c,UnitVector[n,l],UnitVector[n,j]],{l,n}]]

GetNormalizedBracketFlowEquation[c_,i_,j_,k_,n_] :=
c[[i]][[j]][[k]]'==GetBracketFlowEquation[c,i,j,k,n][[2]]
-(c[[i]][[j]][[k]]/(Dot[Flatten[c],Flatten[c]]/2))*Sum[
c[[x]][[y]][[z]]*GetBracketFlowEquation[c,x,y,z,n][[2]],{x,1,n},{y,1,n},{z,1,n}]/2

GetBackwardsBracketFlowEquation[c_,i_,j_,k_,n_] :=
GetBracketFlowEquation[c,i,j,k,n][[1]]==-GetBracketFlowEquation[c,i,j,k,n][[2]]

GetBackwardsNormalizedBracketFlowEquation[c_,i_,j_,k_,n_]:=
c[[i]][[j]][[k]]'==GetBackwardsBracketFlowEquation[c,i,j,k,n][[2]]-
(c[[i]][[j]][[k]]/(Dot[Flatten[c],Flatten[c]]/2))*Sum
[c[[x]][[y]][[z]]*GetBackwardsBracketFlowEquation[c,x,y,z,n][[2]]
,{x,1,n},{y,1,n},{z,1,n}]/2
```

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