The Ricci Bracket Flow

Kevin Kozlowski and Brennan Vincent

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Lie Algebras

To each Lie group there is a corresponding **Lie Algebra**, which is a vector space with a bilinear map $V \times V \rightarrow V$ called the **Lie Bracket**.

Structrue Constants

We can encode the Lie bracket by picking an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ and looking at the so-called **structure constants**, which are the numbers c_{ij}^k in the equation

$$[e_i, e_j] = \sum_{k=1}^{n} c_{i,j}^k e_k$$

Alternatively, we can write the bracket as a bilinear map μ.

$$\mu(e_i, e_j) = \sum_{k=1}^{n} c_{i,j}^k e_k$$

We can describe geometric deformation through a system of differential equations on the structure constants. With respect to these differential equations, the structure constants c_{ij}^k are functions of time. The movement of a bracket through time as defined by these ODEs defines a **flow** on the brackets.

ODEs of the Structure Constants

The flow of the structure constants is described by the equation

$$(c_{ij}^k)' = \sum_{l=1}^{n} (c_{ij}^l \operatorname{Ric}_{kl} - c_{lj}^k \operatorname{Ric}_{li} - c_{il}^k \operatorname{Ric}_{lj})$$

And if the flow is normalized, the equations are of the form

$$(\mu/|\mu|)' = (\text{degree 5 term}) + (\text{degree 3})$$

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Finding Solitons

When studying geometric flow, fixed points are always of interest, as they may represent special or "best" geometries. A **soliton** is a point at which the integral curve through that point is a line through the origin. We can also choose to look at the **normalized flow** given by taking the derivative of $\lambda = \frac{\mu}{|\mu|}|\mu_0|$. Looking at the normalized flow is a way of looking at the integral curves as projected onto a sphere. Note that since solitons are lines through the origin, they are **fixed points** with respect to the normalized flow.

Our motivating example has been the bracket where $\operatorname{ad}_\mu e_i = \mu(e_i, \cdot) = 0 \quad \forall \quad i < 4$ and $\operatorname{ad}_\mu e_4$ is of the form $\operatorname{ad}_\mu e_4 = \begin{pmatrix} c_{4,1}^1 & 0 & 0 & 0 \\ 0 & c_{4,2}^2 & 0 & 0 \\ 0 & 0 & c_{4,3}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The system of differential equations for the structure constants $(c_{ij}^k)'$ is actually solvable in this case, and upon solving it is revealed that this bracket is a solition under the bracket flow. After identifying this soliton, it is natrual to see if there exist any other solitons of similar forms.The brackets investigated were those where $\operatorname{ad}_\mu e_i = 0 \quad \forall \quad i < 4$ and $\operatorname{ad}_\mu e_4$ and special restrictions were placed on the upper-left 2×2 block of $\operatorname{ad}_\mu e_4$ where $\operatorname{ad}_\mu e_4 = \begin{pmatrix} c_{4,1}^1 & c_{4,2}^1 & 0 & 0 \\ c_{4,1}^2 & c_{4,2}^2 & 0 & 0 \\ 0 & 0 & c_{4,3}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Symmetric Case

The first case checked was the one where the upper-left block was of the form

$$\operatorname{ad}_\mu e_4 = \begin{pmatrix} r & s & 0 & 0 \\ s & t & 0 & 0 \\ 0 & 0 & c_{4,3}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

i.e. $c_{4,2}^1 = c_{4,1}^2$. Upon checking using the equations described, we see brackets of this form are also solitons.

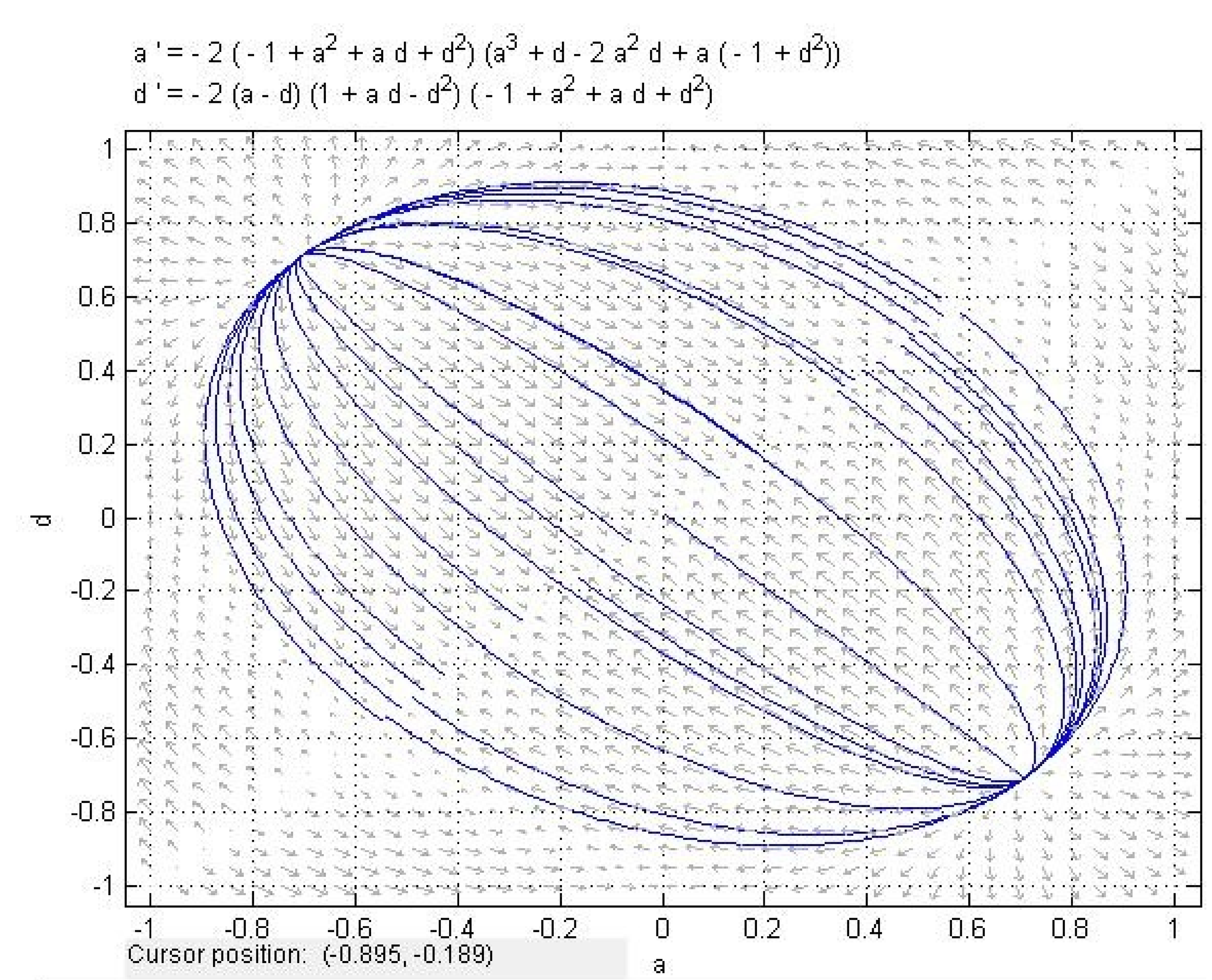
Skew Symmetric plus $r * Id$

Brackets of the form

$$\operatorname{ad}_\mu e_4 = \begin{pmatrix} r & s & 0 & 0 \\ -s & r & 0 & 0 \\ 0 & 0 & c_{4,3}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

also are solitons.

Flow of Non-Soliton Brackets



Here we look at cases where the 2×2 block is of the form $\begin{pmatrix} a & b \\ -b & d \end{pmatrix}$ and the bracket is unimodular (the matrices of $ad_\mu e_n$ have 0 trace). In this specific picture, we fix $|\mu| = 2$ and plot d against a. We can see that the brackets flow towards the line $a = d$. But brackets of this form are solitons when $a = d$ (see the note to the left). Thus in this case, these solitons are attractors.