LOG-LINEAR ODES AND APPLICATIONS TO THE RICCI FLOW FOR HOMOGENEOUS SPACES

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1. Introduction

1.1. **Background.** Given a Riemannian manifold (M, g), recall the definition of Ricci curvature.

Definition 1. Given a point $p \in M$, and a vector $V \in T_pM$, let $\{V = E_1, E_2, ..., E_n\}$ be an orthonormal basis for T_pM. The **Ricci curvature** of V is defined to be

$$\mathrm{ric}_g(V, V) = \sum_{i=2}^{n} \mathrm{sect}(V \wedge E_i).$$

The Ricci curvature averages the sectional curvatures near a point. The *Ricci flow* is a geometric evolution defined by

$$\frac{\partial}{\partial t} g = -2\mathrm{ric}_{g(t)}.$$

It tries to distribute the curvature evenly around a manifold. In doing so, the Ricci flow preserves the symmetries of the space, and, in the limit, can increase the symmetries of the space. It has been extensively studied in [Ham82], [Pera], [Perc], and [Perb] and it was applied most famously to solve the Poincare conjecture.

In general, the Ricci flow is a PDE, however on a homogeneous space it reduces to a system of ODEs. Recall the definition of a homogeneous space.

Definition 2. A **homogeneous space** is a Riemannian manifold with an isometry between any two points.

The advantage of working on a homogeneous space is that the Riemannian structure can be reduced to an inner product on one tangent space. The Ricci flow can be transformed to a flow on the tangent space, reducing the original PDE to a system of ODEs. The Ricci flow on homogeneous spaces has been studied in [IJ92], [KM01], [IJL06] [`?lott07`], [Pay], and [GP]. We will be looking at a related flow called the structure vector flow.

Lie groups with left-invariant metrics form an important subclass of homogeneous spaces. The Ricci flow reduces to a flow on the Lie algebra of the space, however we have a choice of how to implement the flow. One can, as in any other homogeneous case, allow the inner product on the Lie algebra to evolve over time. Another option, called the *structure vector flow*, is to fix the metric and allow the structure constants of the Lie algebra relative to a time-dependent orthonormal basis to evolve over time. The structure vector flow has been studied in [Gal07], [Pay], and [GP]. In the presence of a "stably diagonal basis," on certain Lie algebras including nilpotent and completely solvable Lie algebras, the structure vector flow will take the form of a homogeneous log-linear ODE. We will study the behavior of this broad class of ODEs in order to understand systems coming the structure vector flow.

Our analysis will focus on three aspects of the qualitative behaviour of such systems of ODEs. First, we are interested in finding when solutions will approach the origin. Algebraically, this would mean that the Lie algebra is "flattening out" as time increases, and is approaching the abelian Lie algebra. Next, we are interested in when soliton trajectories occur.

Definition 3. A metric g_0 is a **Ricci soliton** if

$$g(t) = c(t)\phi_t^* g_0$$

solves the Ricci flow for a function $c(t)$ and a one-parameter family of diffeomorphisms $\phi_t : M \to M$.

Because the Ricci flow is invariant under pullbacks by diffeomorphisms and scaling, solitons evolve in a self-similar way, analogous to the motion of a standing wave. As curves in the phase space, solitons are trajectories moving along invariant rays from the origin. We hope to characterize when such solutions exist. Finally, we hope that other solution curves get closer and closer to soliton trajectories in the limit. This would mean that our geometry, as it is flattening out, is also becoming more regular.

1.2. **Summary of results.** Given 2×2 HLL ODEs (see Definition 4) defined by matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

we completely classified the cases where all solutions asymptotically approach the origin, the cases when soliton trajectories exist, and the cases when solutions move towards a soliton ray. For n-dimensional systems, we determined conditions that cause all solutions to approach the origin. Finally, we found necessary and sufficient conditions for the existence of soliton trajectories that approach the origin.

2. Preliminaries

2.1. Log-linear ODEs.

Definition 4. A system of ordinary differential equations is called *log-linear (LL)* if it has the form

$$\begin{aligned} x_1' &= x_1(\alpha_1 + a_{11}x_1 + \cdots + a_{1n}x_n) \\ &\vdots \\ x_n' &= x_n(\alpha_n + a_{n1}x_1 + \cdots + a_{nn}x_n). \end{aligned} \tag{1}$$

It is *homogeneous log-linear (HLL)* if $\alpha_1 = \cdots = \alpha_n = 0$, and *symmetric* if the matrix $A = (a_{ij})$ is a symmetric matrix.

Remark 2.1. Log-linear ODEs have also been studied under the name *generalized Lotka-Volterra equations* in the biological and social sciences. However, the choices of coefficients in those contexts result in qualitatively different solutions than do our choices.

2.2. **ODEs.** Recall the general existence and uniqueness theorem of ordinary differential equations. Because log-linear ODEs are defined by smooth functions, one can always find unique local solutions to the equations given an initial condition. However, in most of our analyses we hope to show existence for longer periods of time. The following definitions are essential for basic phase plane analysis of ODEs.

Definition 5. Given an n-dimensional system of ODEs, a point $a \in \mathbb{R}^n$ is called a *fixed point* if $x_i'(a) = 0$ for all $i = 1, \dots, n$. A *nullcline* is a set $U \subset \mathbb{R}^2$ such that for some i, $x_i' = 0$ on U.

Remark 2.2. In the case of HLL ODEs, nullclines have a simple geometric description. They are the hyperplanes defined by the linear part of the equations.

Definition 6. Given an n-dimensional system of ODEs, let $U \subset \mathbb{R}^n$. Let $\gamma(t)$ be a solution curve intersecting U and let $a \in \mathbb{R}$ be the smallest time such that $\gamma(a) \in \bar{U}$. Furthermore, suppose that for all $t > a, \gamma(t) \in U$. If this holds for all solution trajectories intersecting U, then U is called an *invariant set.*

The following theorem shows that within a compact invariant set, we have stronger results for existence and uniqueness. Additionally, for LL ODEs, the signs within a compact invariant set will often be constant. In this case, the solutions must tend to a fixed point.

Theorem 2.3. *Given an n-dimensional system of ODEs, let $K \subset \mathbb{R}^n$ be a compact invariant set and let $\gamma(t)$ be a solution trajectory intersecting K. Let $a \in \mathbb{R}$ be the smallest time such that $\gamma(a) \in K$. Then:*

(1) *$\gamma(t)$ exists for all time $t \geq a$.*
(2) *If the signs of $x_1', \dots, x_n'$ are constant throughout K, then $\lim_{t\to\infty} \gamma(t)$ exists and is equal to a fixed point of the flow.*

Proof. The proof of (1) is a standard theorem of ODE. See, for example, [NS60].

For (2), let $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$. By (1) each γ_i is defined on $\mathbb{R}$. To show that $\gamma(t)$ approaches a limit as $t \to \infty$, it suffices to show that each γ_i approaches a limit as $t \to \infty$. Let i be arbitrary and suppose WLOG that γ_i is strictly increasing. Since $\gamma(t) \subset K$ for all t in some interval $[a, \infty), \gamma_i$ is bounded above and strictly increasing on $[a, \infty)$. Therefore γ_i approaches a limit as $t \to \infty$, and thus $\lim_{t\to\infty} \gamma(t)$ also exists.

Now we must show that $\lim_{t\to\infty} \gamma(t)$ is equal to a fixed point of the flow. Suppose that $\lim_{t\to\infty} \gamma(t) = \vec{b}$ is not a fixed point of the flow and assume WLOG that $x_1'(\vec{b}) > 0$. Let b_1 be the first component of $\vec{b}$. Because x_1' is continuous with respect to $x_1, \dots, x_n$, and greater than 0 on K, there is a closed ball $\bar{B}_\delta(\vec{b})$ of radius δ centered at $\vec{b}$ such that on $\bar{B}_\delta(\vec{b}), x_1'$ is bounded below by a constant $c > 0$. Fix $s \in \mathbb{R}$ such that $\gamma(t) \in \bar{B}_\delta(\vec{b})$ for all $t \geq s$. Such an s exists because $\lim_{t\to\infty} \gamma(t) = \vec{b}$. Let $d = |\gamma_1(s) - b_1|$. Since γ_1 is strictly increasing on K, we have $\gamma_1(s) < b_1$. Also note that $d \leq \delta$ since both $\gamma_1(s)$ and b_1 are contained in $\bar{B}_\delta(\vec{b})$. Consider $\gamma_1(s + \frac{2d}{c})$. Since x_1' is bounded below by c, we have

$$c \leq \left| \frac{\gamma_1(s + \frac{2d}{c}) - \gamma_1(s)}{\frac{2d}{c}} \right|$$

which implies that

$$\begin{aligned}2d \le |\gamma_1(s+\frac{2d}{c}) - \gamma_1(s)| &= |\gamma_1(s+\frac{2d}{c}) - b_1 + b_1 - \gamma_1(s)| \\ &\le |\gamma_1(s+\frac{2d}{c}) - b_1| + |b_1 - \gamma_1(s)| \\ &= |\gamma_1(s+\frac{2d}{c}) - b_1| + d.\end{aligned}$$

Therefore $|\gamma_1(s+\frac{2d}{c})-b_1| \ge d$. Since γ_1 is strictly increasing, we actually have that $\gamma_1(s+\frac{2d}{c}) > b_1$ and so $\lim_{t\to\infty} \gamma_1(t) > b_1$, contradicting our assumption that $\lim_{t\to\infty} \gamma(t) = \vec{b}$. □

As the previous theorem shows, compact invariant sets are useful tools for analyzing the behavior of LL ODEs. The next theorem gives a way to prove that a set is invariant. Later, these results will form the basis for our phase plane analysis of 2×2 systems of HLL ODEs.

Definition 7. Let $F(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ vector field on $\mathbb{R}^n$ and let P be an $(n-1)$-dimensional hyperplane with normal vector $\vec{v}$. The *pointwise flux* of F through P at a point $p_0 \in P$ with respect to the choice of normal $\vec{v}$ is defined to be

$$F(p) \cdot \vec{v}.$$

We recall a standard result from phase-plane analysis.

Proposition 2.4. *Given an n-dimensional system of ODEs defined by smooth functions, let $F(x_1, \ldots, x_n)$ be the vector field defined by the system of ODEs. Let A be a region bounded by $(n-1)$-dimensional hyperplanes and describe the hyperplanes using $\vec{v}$ the inward pointing normal with respect to A. Suppose that the pointwise flux of F through the boundary of A with respect to $\vec{v}$ is positive at each point. Then A is an invariant set.*

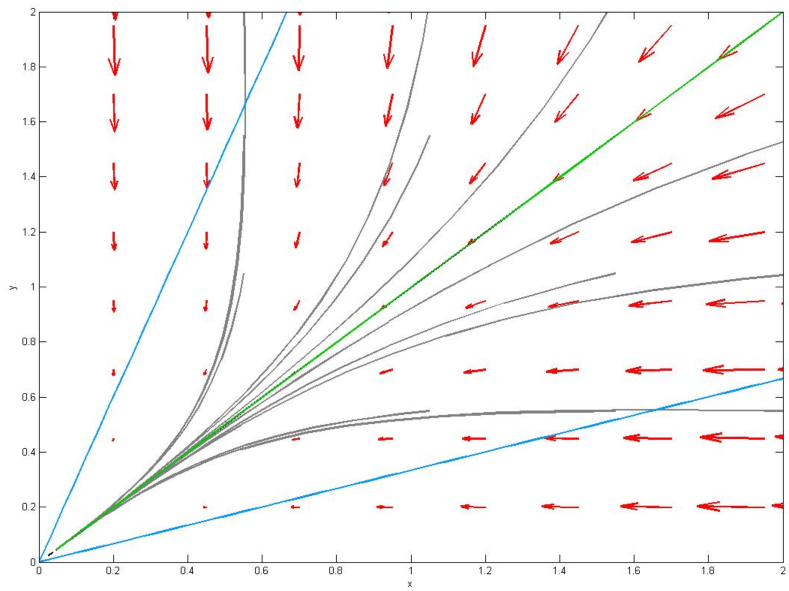
3. Homogeneous Log-linear ODEs

3.1. **Prototypical Examples.** In order to get a feel for systems of HLL ODEs, it will be helpful to examine plots of low dimensional examples.

Example 3.1. Consider the system

$$\begin{aligned}x' &= x(-3x+y) \\ y' &= y(x-3y)\end{aligned}$$

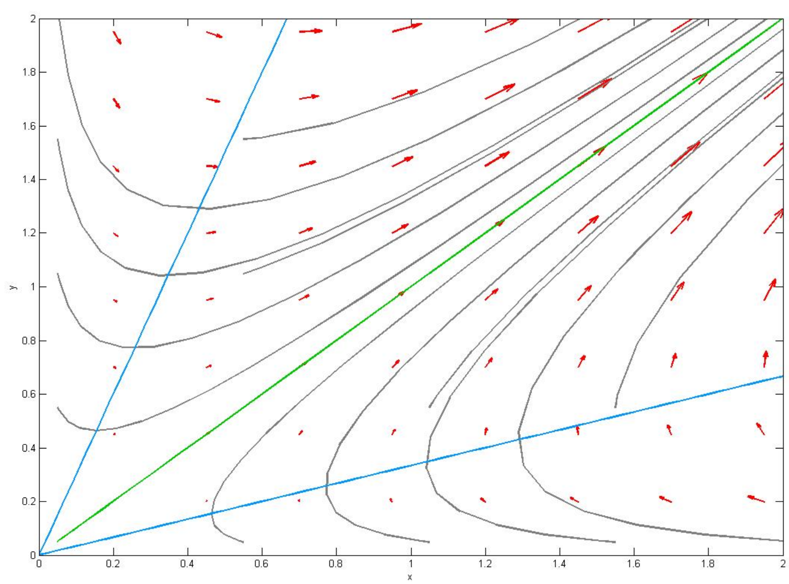
whose graph is given below.



The gray curves are general solutions to the ODEs, the blue lines are the nullclines, and the green line is the soliton trajectory. In this case, all of the solutions are flowing towards the origin and are approaching the soliton ray as $t \to \infty$.

Example 3.2. Consider the system:

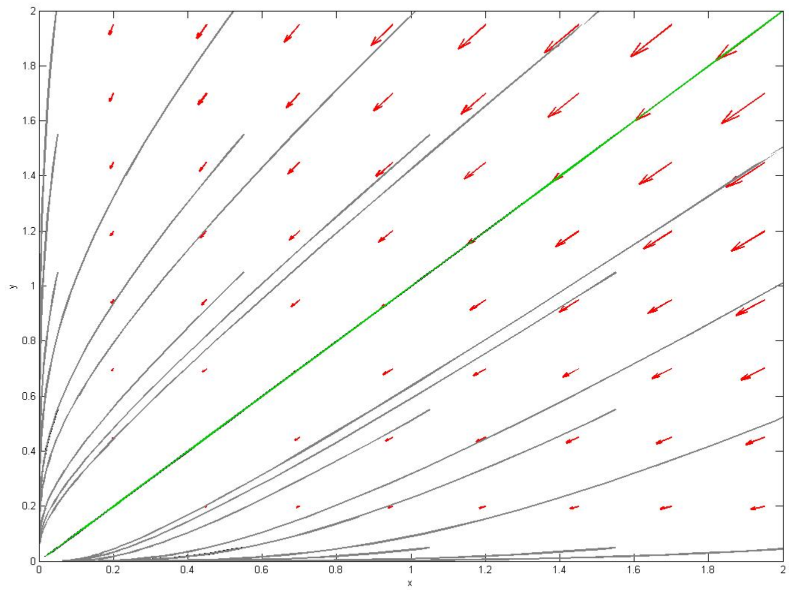
$$x' = x(-x + 3y)$$
$$y' = y(3x - y)$$



In this case, the solution curves are approaching the soliton trajectory, but the flow is directed away from the origin.

Example 3.3. Consider the system

$$x' = x(-x - 4y) \qquad\qquad y' = y(-4x - y)$$



In this case, the solution curves are not approaching the soliton trajectory, but the flow is directed towards the origin.

3.2. **Two Dimensional Phase Plane Analysis.** In order to understand HLL ODEs in n-dimensions, it will help to completely analyze the 2-dimensional case first. First we eliminate some combinations of coefficients that would prevent solutions from going to the origin in the closed first quadrant.

Proposition 3.4. *If the system of symmetric homogeneous log-linear ODEs defined by*

$$x' = x(ax + by)$$
$$y' = y(bx + ay) \tag{2}$$

has the origin as an attractive fixed point for all trajectories in the closed first quadrant, then $a < 0$*.*

Proof. The line $x = 0$ is an x-nullcline. In order for the flow along this nullcline to be directed towards the origin in the first quadrant, $y' < 0$. On this$x$-nullcline, $y' = y(ay)$, therefore $a < 0$. Similarly, on the y-nullcline defined by $y = 0$, we want the flow on this nullcline to be directed towards the origin, so $x' < 0$ and $a < 0$. □

Since a must be negative, we can consider four remaining cases where the solutions may be flowing towards the origin.

Proposition 3.5. *Given the symmetric system of ODEs* (2)*, there are four distinct cases for the qualitative behavior of solutions in the first quadrant.*

- *In case 1: $a < 0$, $\frac{a}{b} > \frac{b}{a}$, and the solution curves approach the origin.*
- *In case 2: $a < 0$, $\frac{b}{a} < \frac{a}{b}$, the solution curves do not approach the origin.*
- *In case 3: $a < 0, b < 0$, and the solution curves approach the origin.*
- *In case 4: $\frac{a}{b} = \frac{b}{a}$, and the solution curves do not approach the origin.*

Proof. In case 1: The boundaries of the set $\{(x, y)|ax + by \geq 0, bx + ay \leq 0\}$ in the first quadrant are the x-nullcline and y-nullcline, which are defined by

$$ax + by = 0,$$

$$bx + ay = 0,$$

respectively. Additionally, $\frac{a}{b} > \frac{b}{a}$. On the y-nullcline, let us calculate the inward pointing normal. The y-nullcline has a positive slope, and in order for the normal to be pointing into the set, the x component must be positive and the y component must be negative. For this to be satisfied, the inward pointing normal must be $(-b, -a)$ since the constraints already placed on a and b are $a < 0$ and $b > 0$. In addition, $x' < 0$ on the y-nullcline. The direction of the flux across the y-nullcline is defined by:

$$(-b, -a) \cdot (x', 0) = -bx' > 0.$$

Therefore the flux is directed in the same direction as the inward pointing normal, so the flux is flowing inwards toward the origin across this boundary. Similarly the direction of the flux across the x-nullcline is defined by:

$$(-a, -b) \cdot (0, y') > 0,$$

where $(-a, -b)$ is the inward pointing normal, and $y' < 0$. Therefore the flux across the entire boundary of this set is always directed inward, so by Proposition 2.4 the set is invariant.

In case 2: The boundaries of the set $\{(x, y)|ax + by \leq 0, bx + ay \geq 0\}$ in the first quadrant are the x-nullcline and y-nullcline, as defined above. Additionally, $\frac{b}{a} > \frac{a}{b}$. On the y-nullcline, the inward pointing normal is defined by (b, a). This is the inward pointing normal, because based on the constraints we have placed on a and b, we want the x component to be positive and the y component negative. Additionally,$x' > 0$. The direction of the flux across the y-nullcline is defined by:

$$(b, a) \cdot (x', 0) > 0.$$

Therefore the flow across this nullcline is directed inward. Similarly, the flow across the x-nullcline is directed inward. Therefore by Proposition 2.4 the set is invariant.

In case 3: $a < 0$ and $b < 0$ and both the x-nullcline and y-nullcline, as defined above lie beneath the first quadrant. Therefore the set in question is the entire first quadrant, and the boundaries are $x = 0$, $y = 0$. These lines are an x-nullcline and y-nullcline, respectively. The flux along this x-nullcline is directed downward towards the origin; see Proposition 3.2. Similarily, the flux along the y-nullcline is directed torwards the origin. By Proposition 2.4, the set is invariant.

In case 4: $\frac{a}{b} = \frac{b}{a}$, which results in an overlap of the x-nullcline and y-nullcline as defined in case 1. In this case, the x and y-nullclines form a line of fixed points. Solution trajectories in the first quadrant flow towards the line of fixed points, and do not all approach the origin. □

3.3. **Existence and Uniqueness of Solutions.** Now we move to results in higher dimensions. Note that all of these theorems, when reduced to the 2-dimensional case, fall out of our analysis in the previous section.

Theorem 3.6. *Suppose that $x_1', \ldots, x_n'$ is a HLL system of ODE, and suppose that $a_{11}, \ldots, a_{nn} < 0$. Then for all points in the region $A = \{\vec{x} | x_1, \ldots, x_n \geq 0\}$, solution trajectories exist for all positive time. Additionally, if $\gamma(t)$ is a solution trajectory then $\lim_{t\to\infty} \gamma(t) = \vec{0}$.*

Proof. We proceed by induction on n. For $n = 1$, the ODE simplifies to

$$x_1' = ax_1^2$$

where $a < 0$. In this case, an explicit solution is given by

$$x(t) = \frac{-1}{a(t+C)}.$$

If we specify an initial condition $x(0) = x_0$, then we can solve for C to obtain $C = \frac{-1}{ax_0}$. Clearly, $\lim_{t\to\infty} x(t) = 0$.

Now consider the n-dimensional case. In each of the hyperplanes $x_i = 0\,(i = 1, \ldots, n)$, the ODE reduces to a system of $n-1$ equations with negative coefficients. By the inductive hypothesis, this implies that $\vec{0}$ is an attractive fixed point for points on the boundary of A. Now we must check that trajectories through points in the interior of A approach the origin. Note that because all the coefficients are negative, each $x_i' < 0$ on the interior of A. The null-hyperplanes defined by the linear part of each x_i' will pass beneath A, and so x_i' will not change sign within A. Fix $p = (p_1, \ldots, p_n)$ in the interior of A. Consider the closed rectangle R formed by the coordinate hyperplanes and the set of hyperplanes defined by $x_i = p_i$. The pointwise flux is 0 on the coordinate hyperplanes. On the hyperplane $x_i = p_i$, the inward pointing normal $\hat{n}_i$ is given by $\hat{n}_i = (0, \ldots, -1, \ldots, 0)$ where -1 appears in the ith coordinate. Thus the pointwise flux is given by

$$\hat{n}_i \cdot (x_1', \ldots, x_i', \ldots, x_n') = -x_i' > 0$$

since $x_i' < 0$ on A. Therefore R is a compact invariant set, and $p \in R$. Thus by Theorem 2.3 there is a solution trajectory $\gamma(t)$ defined for all positive time with $\gamma(0) = p$, and $\lim_{t\to\infty} \gamma(t)$ is a fixed point of the flow. Since the only fixed point contained in R is the origin, $\lim_{t\to\infty} \gamma(t) = \vec{0}$. □

While this gives a positive condition for solutions to approach the origin, it covers only a highly specific subset of HLL ODEs. A more useful positive condition can be found using Lyapunov functions. Recall the definition of a Lyapunov function.

Definition 8. Consider an n-dimensional system of smooth ordinary differential equations. A C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ is a *Lyapunov function* for the flow if it satisfies:

- $f(\vec{0}) = 0$.
- $f(\vec{x}) > 0$ for $\vec{x} \in \mathbb{R} \setminus \{\vec{0}\}$.
- f is decreasing along the flow. More precisely, for any solution trajectory $\gamma(t)$ in $\mathbb{R} \setminus \{\vec{0}\}$, the function $f(\gamma(t))$ is strictly decreasing.

Lyapunov functions are useful for analyzing the qualitative behavior of ODEs because of the following theorem. See [Jac91] for a proof.

Theorem 3.7. *Suppose that there exists a Lyapunov function for a system of ODEs. Then the origin is an attractive fixed point.*

Theorem 3.8. *Given an n-dimensional system of symmetric HLL ODE, let $A = (a_{ij})$. Suppose that there exists a vector $\vec{v} = (v_1, \ldots, v_n)^T$ with $v_1, \ldots, v_n > 0$ such that the product $A\vec{v}$ has all negative entries. Then the origin is an attractive fixed point for trajectories lying in the region $B = \{\vec{x} | x_i \geq 0, i = 1, \ldots, n\}$.*

Proof. Consider the function

$$f(t) = x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n} = \prod_{i=1}^{n} x_i^{v_i}.$$

Because B is an invariant set, it will suffice to show that f is a Lyapunov function for the flow in B. Clearly $f(0) = 0$ and $f > 0$ on B. Now we must show that f is decreasing along the flow. Using logarithmic differentiation, we get

$$\begin{aligned} \frac{f'}{f} = (\log(f))' &= \left(\sum_{i=1}^{n} v_i \log x_i \right)' \\ &= \sum_{i=1}^{n} \frac{v_i x_i'}{x_i} \\ &= \sum_{i=1}^{n} v_i \left(\sum_{j=i}^{n} a_{ij} x_j \right) \end{aligned}$$

and therefore

$$f' = \left(\prod_{k=1}^{n} x_k^{v_k} \right) \left(\sum_{i=1}^{n} v_i \left(\sum_{j=i}^{n} a_{ij} x_j \right) \right).$$

Since $\prod_{k=1}^{n} x_k^{v_k} > 0$ on B, it suffices to show that $\sum_{i=1}^{n} v_i (\sum_{j=i}^{n} a_{ij} x_j) < 0$ on B. This sum can be written as

$$(v_1, \ldots, v_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{v}^T A \vec{x}.$$

But since this is a 1×1 matrix, it is equal to its transpose. This gives

$$\vec{v}^T A \vec{x} = \vec{x}^T A^T \vec{v} = \vec{x}^T A \vec{v} < 0$$

where we have used the facts that A is symmetric, $v_i x_i > 0$ for all i and for all points in B, and $A\vec{v}$ has all negative entries. This completes the proof that f is a Lyapunov function on B, and thus completes the proof that the origin is an attractive fixed point for trajectories in B. □

Remark 3.9. Although Theorem 3.8 does not give necessary and sufficient conditions for solution to approach the origin, it does give a flexible positive condition. Later we will see that the condition from Theorem 3.8 is highly compatible with the condition for the existence of a soliton trajectory.

3.4. **Existence of Soliton Trajectories.** Now that we have some results for when solutions will approach the origin, we will attempt to find the soliton trajectories that we hope our solutions are moving closer to over time.

Theorem 3.10. *Given a system of n HLL ODE, let $A = (a_{ij})$ be the matrix of coefficients for the ODE, and let $B = \{\vec{x}|x_1, \ldots, x_n \geq 0\}$. The ray in B in the direction of a vector $\vec{v} = (v_1, \ldots, v_n)^T$ is a soliton trajectory if and only if $\vec{v}$ is a solution to the equation*

$$A\vec{v} = \lambda \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

for some $\lambda < 0$. Furthermore, an explicit solution for the soliton trajectory is given by

$$x_i(t) = \frac{-v_i}{\lambda(t + C_i)}$$

where C_i is determined by the initial conditions on the soliton ray. If $x_i(0) = x_0^i$ then $C_i = \frac{-v_i}{\lambda x_0^i}$.

Proof. First suppose that the ray in the direction $(v_1, \ldots, v_n)$ is a soliton trajectory. Then the solutions $x_1, \ldots, x_n$ satisfy the relationship

$$\frac{x_1}{v_1} = \frac{x_2}{v_2} = \cdots = \frac{x_n}{v_n} \tag{3}$$

and similarly the derivatives must satisfy

$$\frac{x_1'}{v_1} = \frac{x_2'}{v_2} = \cdots = \frac{x_n'}{v_n}. \tag{4}$$

Using (1), the system of ODEs will decouple into the system

$$\begin{aligned} x_1' = x_1^2 \left(\sum_{i=1}^{n} \frac{v_i a_{1i}}{v_1} \right) &= z_1 x_1^2 \\ \vdots \qquad\qquad & \quad \vdots \\ x_n' = x_n^2 \left(\sum_{i=1}^{n} \frac{v_i a_{ni}}{v_n} \right) &= z_n x_n^2 \end{aligned}$$

where $z_j = \sum\limits_{i=1}^{n} \frac{v_i a_{ji}}{v_j}$. Equation (2) will place some constraints on the z_j's. For example, since $\frac{x_i'}{v_i} = \frac{x_j'}{v_j}$, we have that

$$\frac{z_i x_i^2}{v_i} = \frac{x_i'}{v_i} = \frac{x_j'}{v_j} = \frac{z_j x_j^2}{v_j}.$$

Using (2) to substitute for x_j^2, we get

$$\frac{z_i x_i^2}{v_i} = \frac{z_j \left(\frac{v_j x_i}{v_i}\right)^2}{v_j} = \frac{z_j v_j x_i^2}{v_i^2}$$

which implies that

$$\begin{vmatrix} z_i & \frac{1}{v_i} \\ z_j & \frac{1}{v_j} \end{vmatrix} = \frac{z_i}{v_j} - \frac{z_j}{v_i} = 0.$$

Since this holds for all i, j, we must have that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \lambda \begin{pmatrix} \frac{1}{v_1} \\ \vdots \\ \frac{1}{v_n} \end{pmatrix}$$

for some $\lambda \neq 0$. We can now rewrite the decoupled system as

$$x_1' = x_1^2 \left(\sum_{i=1}^{n} \frac{v_i a_{1i}}{v_1} \right) = \frac{\lambda}{v_1} x_1^2$$

$$\vdots$$

$$x_n' = x_n^2 \left(\sum_{i=1}^{n} \frac{v_i a_{ni}}{v_n} \right) = \frac{\lambda}{v_n} x_n^2.$$

By cancelling the term $\frac{x_i^2}{v_i}$ from each side of each of these equations, we get the relationship

$$A\vec{v} = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix}.$$

Now it suffices to show that $\lambda < 0$. Because the system completely decouples, we need only solve for each x_i individually. Since

$$x_i' = \frac{\lambda}{v_i} x_i^2,$$

separation of variables yields the solution

$$x_i(t) = \frac{-v_i}{\lambda(t + C_i)}$$

for some constant C_i that depends on the initial condition. To solve for C_i, simply plug in the value x_0^i at $t = 0$ to obtain $C_i = \frac{-v_i}{\lambda x_0^i}$. Because $v_i > 0$, we must have that $\lambda < 0$ so that $x_i > 0$ for all time $t \in (-C, \infty)$. This ensures that the trajectory is contained entirely in B. □

Remark 3.11. In order to find the soliton trajectory for an explicit set of equations, simply set $\lambda = -1$ and solve the equation

$$A\vec{v} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}.$$

Since λ is determined only by the magnitude of $\vec{v}$ and not its direction, this will always yield a vector in the direction of the soliton ray.

Corollary 3.12. *Whenever a soliton trajectory is present, solutions in the first quadrant will approach the origin.*

Proof. This follows directly from Theorem 3.8 and Theorem 3.10. □

This corollary is both surprising and encouraging. If a soliton trajectory is present and moving towards the origin, all other solutions will also be moving towards the origin.

3.5. **Stability.** Through our analysis, we hoped to characterize when solution trajectories approach the origin, when soliton trajectories exist, and when solutions get closer to the soliton in the limit. We have achieved our first two goals: in the presence of a soliton, all solutions in the first quadrant move towards the origin. What remains is to find conditions for when solutions will move towards the soliton as time increases.

Theorem 3.13. *Given a system of* 2 *HLL ODE, let* $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ *be the matrix of coefficients for the ODE. If* $b > a$*, then solution trajectories approach soliton trajectories.*

Proof. Let $\theta(t)$ be the angle between the solution trajectory and the soliton trajectory. Let $\vec{v}$ be the soliton trajectory and let $x(t) = x_1(t), x_2(t)$. Then

$$\cos^2\theta(t) = \left(\frac{\vec{v}\cdot\vec{x}(t)}{||x||||\vec{v}||}\right)^2.$$

It suffices to show that $\cos\theta(t)$ is increasing to 1 as time approaches infinity. Therefore we want the derivative of $\cos\theta(t)$ to be positive. The derivative of $\cos(\theta(t))$ is given by

$$(\cos\theta(t))' = \left(\frac{\vec{v}\cdot\vec{x}(t)}{||x||||\vec{v}||}\right)' = 2\frac{\vec{v}\cdot\vec{x}(t)}{||x||}\frac{||x||||\vec{v}||(\vec{v}\cdot\vec{x}(t))' - ||x||'||\vec{v}||'(\vec{v}\cdot\vec{x}(t))}{||x||^2||\vec{v}||^2}.$$

Because we are working in the first quadrant, $x(t) > 0$ and $\vec{v}$ is assumed to be positive. Therefore the sign of this derivative is dependent on the following equation:

$$\begin{aligned}(\cos\theta(t))' &= (v_1x_1' + v_2x_2')(x_1^2 + x_2^2) - (v_1x_1 + v_2x_2)(x_1x_1' + x_2x_2') \\ &= v_1x_1'x_2^2 + v_2x_2'x_1^2 - v_1x_2'x_1x_2 - v_2x_1'x_1x_2.\end{aligned}$$

Substituting

$$\begin{aligned}x_1' &= x_1(ax_1 + bx_2) \\ x_2' &= x_2(bx_1 + ax_2),\end{aligned}$$

we obtain

$$\begin{aligned}(\cos\theta(t))' &= v_1x_2^2(ax_1 + bx_2) + v_2x_1^2x_2(bx_1 + ax_2) \\ &\quad - v_1x_1x_2^2(bx_1 + ax_2) - v_2x_1^2x_2(ax_1 + bx_2) \\ &= (x_1x_2)[v_1a(x_1x_2) + v_1b(x_2^2) - v_1b(x_1x_2) \\ &\quad - v_1a(x_2^2) + v_2b(x_1^2) + v_2a(x_1x_2) - v_2a(x_1^2) - v_2b(x_2x_1)].\end{aligned}$$

By Theorem 3.10

$$(x_1x_2)[v_1(b-a) - v_2(a-b)] = 0$$

Therefore in order for $(\cos\theta(t))' > 0$ it suffices to check

$$\begin{aligned}0 &< v_1x_2^2(b-a) - v_2x_1^2(a-b) \\ &= (b-a)(v_1x_2^2 + v_2x_1^2),\end{aligned}$$

but this is true since $b > a$ by assumption.

□

Although Theorem 3.13 solves the problem for a special subset of 2-dimensional systems, we have not been able to solve the problem in all dimensions. The same methods may extend to higher dimensions, however finding the correct conditions for the angle to be a decreasing function under the flow is much harder when more variables are introduced.

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